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On Geometric Partial Differential Equations and Contact Transformations

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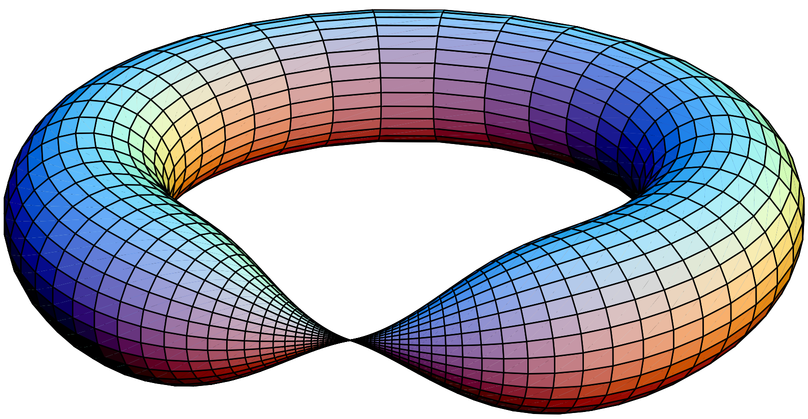

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Abstract

In this thesis we present a geometric formulation of the classical partial differential equation. In chapter 1 and 2 we define the first and second order contact bundle of a smooth manifold. The contact bundles are the natural spaces on which we can define partial differential equations. We also define contact transformations which are a generalization of coordinate transformations.

In chapter 3 we apply our theory to give a description of Monge-Ampère equations with constant coefficients. In the last chapter we present some possibilities for future research in this field.

Notation and conventions

In this text we work most of the time over the real numbers. Unless stated otherwise all functions are real-valued and all vector spaces are real vector spaces. All functions and manifolds are assumed to be smooth, i.e. of class C^∞, unless stated otherwise.

A multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ is an element of $\mathbb{Z}^n_{\geq 0}$ for an integer $n \geq 1$. From the context the value of n will be clear. We write $|\alpha|$ for the length of the multi-index, it is defined as $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$. In the text multi-indices will be used to write partial derivatives in a compact notation. For a function $u(x)$ of the n variables $x_1, \dots, x_n$ we write

$$\partial^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x^\alpha} = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} u.$$

On set of the multi-indices we have a natural ordering induced from the natural ordering on $\mathbb{Z}^n$ (the lexicographic ordering). If N is the number of multi-indices with $0 \leq |\alpha| \leq r$, then we write

$$(\partial^\alpha u(x))_{|\alpha| \leq r}$$

for the vector $(\alpha_1, \dots, \alpha_N) \in \mathbb{R}^N$, where $\alpha_1, \dots, \alpha_n$ are the elements of the set $\{ \alpha \in \mathbb{Z}^n_{\geq 0} \mid |\alpha| \leq r \}$ in the natural ordering. For example for $n = 2, r = 1$ we have

$$((\partial^\alpha u)(x))_{|\alpha| \leq r} = (u(x), \partial_1 u(x), \partial_2 u(x))$$

and for $n = 1, r = 3$ we have

$$((\partial^\alpha u)(x))_{|\alpha| \leq r} = (u(x), \partial u(x), \partial^2 u(x), \partial^3 u(x)).$$

Suppose E is three dimensional vector space with coordinates (X, Y, Ξ). Let C be the linear subspace of E defined by the relation $X = \xi Y$. Then C is a two dimensional subspace of E. We can define an isomorphism ϕ from $\mathbb{R}^2$ to C by taking as coordinates for $\mathbb{R}^2$ the coordinates (X, Ξ) and taking for ϕ the map

$$\phi : \mathbb{R}^2 \to C : (X, \Xi) \to (X, \xi X, \Xi).$$

In the text we will use this construction often and for a given vector space E and relation defining a subspace C we will say that we identify C with the (X, Ξ)-space. This identification will always mean that we identify C with $\mathbb{R}^m$ for a suitable m using a map similar to ϕ above.

When using summations we will often omit the summation boundaries if these boundaries are already clear from the context. We will write for example, $\sum_j x_j y_j$ instead of $\sum_{j=1}^n x_j y_j$, if no confusion is possible about the range $1, \dots, n$ over which the index j is summed.

The end of a remark or example is marked with a $\oslash$, the end of a proof is marked with a $\Box$.

Contents

Introduction

This master thesis is about a geometric framework in which we can describe partial differential equations. In this introduction we give a motivation for introducing this different view on partial differential equations and an outline of the different topics treated is this thesis.

This thesis was written such that it is readable for any mathematics student who has knowledge of differentiable manifolds, some differential geometry and the very basics of partial differential equations. In appendix A and B there is a summary of the theory on vector subbundles and symplectic spaces that is essential to the theory in this thesis.

Motivation A second order ordinary differential equation in one variable can often be written as

$$f(x, u, u', u'') = 0 \tag{1}$$

for a smooth function f and suitable initial conditions for $u(0)$ and $u'(0)$. The differential equation is a function f on the space of variables $(x, u, \xi, h) = (x, u, u', u'')$ and the solutions of the equation are functions $u(x)$. We want to describe differential equations (in particular partial differential equations) and the solutions in a more geometric way. In the following paragraphs we give some motivation for this.

For a function $u(x)$ we define the graph of u as the set $\{ (x, y) \in \mathbb{R}^2 \mid y = u(x) \}$. If u is a smooth function, then the graph of u is a smooth submanifold of $\mathbb{R}^2$. Now consider the function $y = f(x) = \sqrt[3]{x}$. This function is continuous, but not differentiable at $x = 0$ and therefore not smooth. The graph of f, however is a smooth submanifold of $\mathbb{R}^2$. This can be seen for example by inverting the function f and writing x as a function of y. The result is $x = f^{-1}(y) = y^3$ and this is a perfectly smooth function. The reflected graph of f is equal to the graph of f^{-1} and is a smooth submanifold of $\mathbb{R}^2$. Notice that by switching the role of the variables x and y we have 'removed' a singularity of the function f but at the same time still have the same object. From this example we can learn that a codimension one smooth manifold is slightly more general the graph of a smooth function, although both concepts are closely related. When working with submanifolds, we can forget some of the problems that arise when working with functions.

The differential equation $f = 0$ is a relation between the function value $u(x)$, the first order derivative $u'(x)$ and the second order derivative $u''(x)$ at x. When we think of the solutions $u(x)$ of this equation as smooth submanifolds of the (x, y)-space, then we can also see this equation as a relation between geometrical properties of the submanifold $\operatorname{graph} u$. For example the coordinates $(x, u(x), u'(x), u''(x))$ can be replaced by the combinations $z(x) = (x, u(x))$, $T(x) = \mathbb{R}(1, u'(x))$ and $C(x) = u''(x)/(1 + u'(x)^2)^{3/2}$. Then $z(x)$ is a point on the graph of u, $T(x)$ is the tangent space of the graph at the point $z(x)$ and $C(x)$ is the curvature of the graph at $z(x)$. A partial differential equation in more independent variables or of higher order also expresses a relation between the geometrical properties of the graph of a solution of the partial differential equation.

The solutions of a differential equation are unchanged if the function f defining the equation is multiplied by a non vanishing function. Therefore a partial differential equation is already determined by the zero set $M = \ker f$ of the function f. In other words, the equation (1) is equivalent to $(x, u(x), u'(x), u''(x)) \in M$. If the function f satisfies certain non-degeneracy conditions, this zero set is a smooth hypersurface in the (x, u, ξ, h)-space. A function u is a solution of the equation if its prolonged graph is contained in the zero set M. For this reason, finding functions u that satisfy the partial differential equation is related to finding submanifolds of the zero set M that are of the type $\{ (x, u(x), u'(x), u''(x)) \mid x \in I \}$, where u is a smooth function and I an open interval.

Notice that in the previous example the differential equation was an equation on the (x, y, ξ, h)-space, with the identification $y = u(x)$, $\xi = u'(x)$ and $h = u''(x)$. In general a partial differential equation of order r in n variables is a function on the 'space of derivatives up to order r'. A more formal definition of this space will be given in terms of the jet bundle of a manifold in section 1.1. There we will also allow functions on a general manifold X and not only functions on subsets of $\mathbb{R}^n$. This generalization also comes from the idea that a specific choice of coordinates is unwanted, and therefore general manifolds must be allowed.

The considerations above have been known for a very long time. One of the first mathematicians, and certainly the most important, that started looking at differential equations in a more geometric fashion was S. Lie (for a short bibliography see [24]). Many ideas in this thesis are originally from his articles or books. The work of Lie was not very well understood in his time and people still have difficulties understanding the arguments and reasoning in his works. Most of the theory in this thesis is related to [11] (which is unpublished at the moment). Despite these difficulties, the ideas of Lie and the work of other mathematicians such as Frobenius, Bäcklund, Cartan, Kähler, Vessiot and others, have led to many interesting results. These results, which we cannot present here, are of course another important motivation for studying the geometry of partial differential equations.

Outline In chapter 1 and 2 we introduce the concept of a contact bundle. This is a generalization of the jet bundles, which are the spaces on which we can define partial differential equations. The contact bundles allow us to consider graphs of functions as the main object of study and not the functions itself. This agrees with the idea in the previous paragraphs that the graph of a function is sometimes more fundamental and easier to work with than the function itself. In these sections we also describe how the concept of a partial differential equation can be translated to this new setting and how we can generalize coordinate transformations to contact transformations. A contact transformation is a coordinate transformation not only between the independent variables, but also between the function value and the derivatives. In chapter 3 we apply this theory to a special type of second order partial differential equations, the Monge-Ampère equations. For these equations the action of contact transformations in one fiber, can be completely calculated and is therefore a nice example of the power of contact transformations. In the last chapter we describe some possibilities for further research.

Acknowledgements Writing a master thesis is not an easy task and I would like to thank everybody who helped me during the past year.

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Chapter 1

The first order contact bundle

1.1 The jet bundle

We now give a more general and more precise description of r-jets of functions and jet bundles. The idea for this jet bundle was already mentioned in the introduction: we want to have a more rigorous definition of the 'space of derivatives'. Although it is quite natural when working with partial differential equations to consider these spaces, a formal definition was only given in 1951 by C. Ehresmann in [12]. The results and definitions in this section are directly related to the original formulation of Ehresmann. The presentation here is similar to that in [7], but is much less detailed in order to keep the theory simple.

Let X be a smooth manifold and assume local coordinates $x_1, \ldots, x_n$ on X have been introduced. We will define the sequence of all derivatives up to order r as an equivalence relation between functions on X. If $f = f(x)$ is a function on X and α is a multi-index, then we write $\partial^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$ for the higher order derivatives of f.

Definition 1.1.1. Let f, g be real valued smooth functions on X. We say that f and g have the same *r-jet* at the point $x \in X$ if in local coordinates $x_1, \ldots, x_n$ for X we have that $(\partial^\alpha f)(x) = (\partial^\alpha g)(x)$ for all multi-indices α with $|\alpha| \leq r$. We write $j^r_x f$ for the equivalence class of r-jets of a function f at the point x. The set of all r-jets at the point x is denoted by $j^r_x(X)$. The r-jet bundle of X is defined as $J^r(X) = \cup_{x \in X} j^r_x(X)$.

Remark 1.1.2 The definition above of an r-jet is independent from the specific choice of local coordinates. The proof follows straightforward from the chain rule for differentiation, for the details we refer to [7, Section 1.3]. ⊘

For any map T the relation $f \sim g \Leftrightarrow T(f) = T(g)$ is an equivalence relation. For $T_x(f) = ((\partial^\alpha f)(x))_{|\alpha| \leq r}$ we find the equivalence relation of the jet bundle of order r. The map T_x induces a map $\tilde{T}_x$ on the r-jets at the point x by $\tilde{T}_x(j^r_x f) = T_x(f)$. In turn this map induces a map $\tilde{T}$ on the jet bundle that leads to local coordinates on $J^r(X)$.

Theorem 1.1.3. *The r-jet bundle of a smooth manifold X is a smooth vector bundle over X. A choice of local coordinates $x_1, \ldots, x_n$ for X induces local coordinates on $J^r(X)$ by*

$$\tilde{T}(j^r_x u) = (x, (\partial^\alpha u(x))_{|\alpha| \leq r}).$$

The projection $J^r(X) \to X$ is given in these local coordinates by $(x, (\partial^\alpha u(x))_{|\alpha| \leq r}) \mapsto x$. The dimension of $J^r(X)$ equals $\dim X + \binom{\dim X + r}{r}$.

Proof. See [7] section 1.3. □

Because of the local coordinates for X in the previous theorem, the r-jet bundle of a manifold X is a formalization of the space of derivatives up to order r of functions on X.

Example 1.1.4 For every smooth manifold X and smooth function f on X we have $J^0(X) = X \times \mathbb{R}$ and $j^0 f : X \to J^0 X : x \mapsto (x, f(x))$. The image of the 1-jet of f is equal to the graph of f. $\oslash$

Example 1.1.5 The 2-jet bundle of $\mathbb{R}^2$ is an 8 dimensional manifold. The fibers j^2_p over the point $p \in \mathbb{R}^2$ are 5 dimensional. The coordinates (x, y) for $\mathbb{R}^2$ induce global coordinates x, y, z, p, q, r, s, t on the jet bundle $J^2(\mathbb{R}^2)$ with

$$x(j^2_{(x,y)}f) = x, \quad y(j^2_{(x,y)}f) = y, \quad z(j^2_{(x,y)}f) = f(x),$$
$$p(j^2_{(x,y)}f) = \frac{\partial f}{\partial x}(x, y), \quad p(j^2_{(x,y)}f) = \frac{\partial f}{\partial y}(x, y),$$
$$r(j^2_{(x,y)}f) = \frac{\partial^2 f}{\partial x^2}(x, y), \quad s(j^2_{(x,y)}f) = \frac{\partial^2 f}{\partial x \partial y}(x, y), \quad t(j^2_{(x,y)}f) = \frac{\partial^2 f}{\partial y^2}(x, y).$$

Note that $\frac{\partial^2 f}{\partial x \partial y}(x, y) = \frac{\partial^2 f}{\partial y \partial x}(x, y)$, since the partial derivatives commute for smooth functions. $\oslash$

For a function $u : X \to \mathbb{R}$ we have defined the r-jet of u at the point x as an equivalence relation between functions on X. We define the r-jet $j^r u$ of a function $u : X \to \mathbb{R}$ to be the function $X \to J^r(X) : x \to j^r_x u$. The r-jet of the function u is a section of the r-jet bundle of X. In local coordinates the r-jet of the function u is obtained by adjoining to $u(x)$ the derivatives up to order r.

With these definitions we can identify the differential equation (1) with a function f on the jet bundle $J^2(\mathbb{R})$ (the bundle $J^2(\mathbb{R})$ is isomorphic to $\mathbb{R}^4$). A function u is called a solution of the differential equation if $f \circ j^2 u$ is identically zero. The initial conditions can also be formulated in terms of the 2-jet of u in a straightforward way. Also partial differential equations can be identified with functions on the jet bundle in this way.

Example 1.1.6 The jet bundle $J^1(\mathbb{R}^2)$ is 5-dimensional and has coordinates x, y, u, u_x and u_y. A function f on $J^1(\mathbb{R}^2)$ defines a partial differential equation. A function $u : \mathbb{R}^2 \to \mathbb{R}$ is a solution of the partial differential equation if $f \circ j^1 u = 0$, i.e.

$$f(x, y, u(x, y), u_x(x, y), u_y(x, y)) = 0, \quad \text{for all } x, y \in \mathbb{R}.$$

$\oslash$

Remark 1.1.7 We can define more general jet bundles and jets of functions by considering mappings $M \to N$ between smooth manifolds. Note that here we have only considered the special case $N = \mathbb{R}$. In this more general setting we can relate differential geometric constructions to the jet-bundles. For example we can define the cotangent space $T^*_m M$ at a point $m \in M$ as $J^1(M, \mathbb{R}) = J^1(M)$ by identifying an element $j^1_m f \in J^1_m M$ with the differential df_m in the cotangent space $(T_m M)^*$. This identification is independent of the function f chosen to represent the jet $j^1_m f$. $\oslash$

1.2 The first order contact bundle and contact structure

In the introduction and the previous section we have seen two concepts that are important when describing functions and partial differential equations: the jet bundle, which is a formalization of the space of variables and the graph of a function. A generalization of these two concepts can be formulated in terms of the contact bundle. In this section we will define the first order contact bundle of a smooth manifold and a vector subbundle of the tangent space of this contact bundle called the contact structure. We will see that this contact bundle is a natural generalization of the jet bundle from the previous section. We will also show how we can reformulate a first order partial differential equation in terms of the first order contact bundle.

For an $(n+1)$-dimensional manifold Z let $G_n(T_zZ)$ be the space of all n-dimensional linear subspaces of T_zZ. There is a natural identification of $G_n(T_zZ)$ with the projectivization of the cotangent space $(T_zZ)^*$. An element of this projectivization $P(T_zZ)^*$ is a nonzero element of $(T_zZ)^*$ modulo a nonzero scalar. We identify $P(T_zZ)^*$ with $G_n(T_zZ)$ by the map $\xi \in P(T_zZ)^* \mapsto \ker \xi \in G_n(T_zZ)$. From now on we identify these two spaces with the above map.

Definition 1.2.1. Let Z be an $(n+1)$-dimensional manifold. The spaces $G_n(T_zZ)$, or equivalently $P(T_zZ)^*$, form a smooth fiber bundle over Z which we denote by

$$P = G_n(TZ) = P(T^*Z)$$

and is called the *first order contact bundle* of Z. The canonical projection $P \to Z$ is denoted by π_Z^P or just π if no confusion is possible.

For $H \in G_n(T_zZ)$ we can write the corresponding element in P as $p = (z, H)$. The base point z is tagged along as a reminder of the tangent space T_zZ in which H lives. We will often write H, when we mean the corresponding element (z, H) in P. In this way an n-dimensional linear subspace of T_zZ, a covector in $(T_zZ)^*$ modulo a nonzero scalar and an element of the first order contact bundle are all identified.

If S is a smooth codimension 1 submanifold of Z, we can embed S in P by sending each point $s \in S$ to its tangent space. More precisely, the map

$$\mathcal{T} : S \to P : s \mapsto (s, T_sS) = T_sS$$

is a smooth embedding of S into P. The map $\mathcal{T}$ is a section of the restriction to S of the bundle $\pi : P \to Z$, in the sense that $\pi \circ \mathcal{T} = \mathrm{id}|_S$. This implies that the projection of $U = \mathcal{T}(S)$ is again equal to S. The manifold U is called the prolongation of S. In this way every smooth n-dimensional submanifold of Z corresponds in a natural way to a smooth submanifold of P. We want to know which of the submanifolds of P correspond in this way to submanifolds in Z. We will now define a structure (called the contact structure) in the tangent bundle of P that will indicate whether a submanifold in P is induced locally by a submanifold or not.

Definition 1.2.2. If $p = (z, H) \in P$, where H is an element of $G_n(T_zZ)$, then we define

$$C_H = C_p = (T_p\pi)^{-1}(H) \subset TP. \tag{1.1}$$

The C_H, $H \in P$ form a smooth vector subbundle C of TP with $\dim C_H = 2n$. The subbundle C is called the (first order) contact structure of P.

When we speak of the first order contact bundle P of a manifold Z we will often mean the pair (P, C) of the contact bundle and the corresponding contact structure defined above.

Because we have defined the contact structure C in an abstract way it is not directly clear why we need this structure. To make this clear we will first give a description of the contact structure and the first order contact bundle in local coordinates. We can always find local (and sometimes even global) coordinates $z = (z_1, \ldots, z_{n+1})$ for Z. These coordinates induce coordinates in the tangent space TZ and coordinates $\eta_1, \ldots, \eta_{n+1}$ in the cotangent space $(T_zZ)^*$. By passing to the projective coordinates $[\eta]$ for $P(T_zZ)^*$, we find coordinates $(z, [\eta])$ for the first order contact bundle P. It is not always convenient to work with the projective coordinates $[\eta]$ and therefore we will often use the standard coordinates for projective space instead of the projective coordinates. In the case of the first order contact bundle this leads to the following situation. Suppose $p = (z, H_0)$ is an element of P. We can always find coordinates $x_1, \ldots, x_n, y$ for Z such that the vector field

$$
\begin{tikzcd}
P \ni p \arrow[r, mapsto] \arrow[d] & C_p \subset TP \arrow[d] \\
Z & TZ
\end{tikzcd}
$$

Figure 1.1: The first order contact structure

$\partial_y = \frac{\partial}{\partial y}$ is transversal to H_0. The vector fields $\partial_{x_1}, \dots, \partial_{x_n}, \partial_y$ form a basis for the tangent space of Z and we write $(X_1, \dots, X_n, Y)$ for the tangent vector $X^1\partial_{x_1} + \dots + X^n\partial_{x_n} + Y\partial_y$. Every H in $G_n(T_zZ)$ that is transversal to Y, i.e. $Y \notin H$, can be written as

$$H = \{\, (X_1, \dots, X_n, Y) \in T_zZ \mid Y = \sum_{j=1}^{n} \xi_j X_j \,\} \tag{1.2}$$

for unique $1 \leq \xi_j \leq n$. The linear subspaces H that are transversal to Y form an open neighborhood of H_0 in $G_n(T_zZ)$. So after a choice of coordinates for Z we can write all $(z, H) \in P$ that are transversal to Y using the $2n+1$ coordinates $(x_1, \dots, x_n, y, \xi_1, \dots, \xi_n)$. In particular the space of elements in $G_n(T_zZ)$ transversal to H is described by the local coordinates $(\xi_1, \dots, \xi_n)$. The relation between the ξ-coordinates and the projective coordinates $[\eta]$ introduced before is given by $\xi_j = -\frac{\eta_j}{\eta_{(n+1)}}$, $1 \leq j \leq n$.

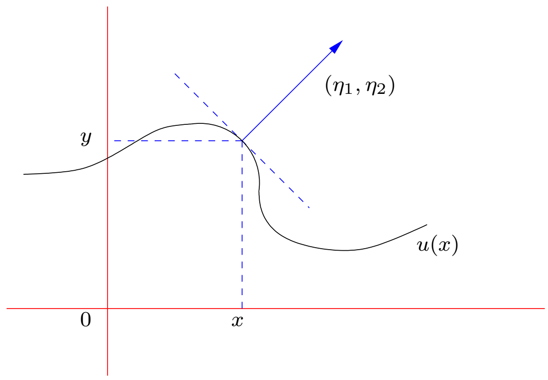


Figure 1.2: The first order coordinates of the graph of a function $u(x)$ in the first order contact bundle $(n = 1)$.

The tangent space T_pP has local coordinates $X_1, \dots, X_n, Y, \Xi_1, \dots, \Xi_n$. We identify these local coordinates with the tangent vector $X_1\partial_{x_1} + \dots + X_n\partial_{x_n} + Y\partial_y + \Xi_j\partial_{\xi_j}$. The definition of C_H by formula (1.1) can be translated to these local coordinates. Note that the tangent map of the projection π is given in the local coordinates by

$$T_H\pi : T_HP \to T_zZ : (X_1, \dots, X_n, Y, \Xi_1, \dots, \Xi_n) \mapsto (X_1, \dots, X_n, Y)$$

This means that if $H \in G_n(T_zZ)$ corresponds to the coordinates $(\xi_1, \dots, \xi_n)$ then

$$\begin{aligned} C_H &= \{\, (X, Y, \Xi) \in T_pP \mid \pi(X, Y, \Xi) \in H \,\} \\ &= \{\, (X_1, \dots, X_n, Y, \Xi_1, \dots, \Xi_n) \in T_pP \mid Y = \sum_j \xi_j X_j \,\}. \end{aligned}$$

The vector subbundle C_H is a codimension one subbundle and can also be described as the kernel of a 1-form ω. The contact structure is given in terms of the differential forms as $C_p = \ker \omega_p$.

The form ω is only defined up to a nonzero scalar and in general ω is only defined locally. Any such 1-form ω is called a first order contact form of the contact structure on P. For the local coordinates introduced above, there is a standard contact form, given by the Pfaffian form $\omega = dy - \sum_j \xi_j dx^j$. In order to simplify notation we will treat the coordinates ξ as a column vector and write $\omega = dy - \xi^T dx$.

In the following theorem we use the concept of an integral manifold. For a definition of an integral manifold and related concepts we refer to appendix A.

Theorem 1.2.3. *Let Z be a $(n+1)$-dimensional manifold with first order contact bundle P and contact structure C.*

i) If S is an n-dimensional submanifold of Z then $U = \mathcal{T}(S)$ is an n-dimensional integral manifold of C. The projection $\pi|_U$ is the inverse of $\mathcal{T}$ and both $\pi|_U$ and $\mathcal{T}$ are diffeomorphisms.

ii) Suppose U is an n-dimensional integral manifold of C. If the projection $\pi : P \to Z$ induces an embedding of U in Z, then $S = \pi(U)$ is a smooth submanifold of Z and $U = \mathcal{T}(S)$.

Proof. See [11, Lemma 2.1]. □

The theorem gives an answer to the question which submanifolds of Z correspond to submanifolds of P using the map $\mathcal{T}$. A smooth submanifold of P must be an integral manifold of the contact structure in order to correspond to a submanifold of Z. Besides this integral condition we also need that the projection of the manifold to Z is a diffeomorphism. In section 1.3 we will see why the projection of a smooth integral manifold of C can fail to be a smooth submanifold of Z.

Every function $u : X \to \mathbb{R}$ defines a smooth submanifold S of Z by taking S equal to the graph of U and therefore a smooth integral submanifold $U = \mathcal{T}(S)$ of P. The following lemma gives us a relation between the jet of this function in the 1-jet bundle of X and the image under $\mathcal{T}$ of the graph of u in the first order contact bundle. It turns out that in local coordinates the manifold $U = \mathcal{T}(S)$ is exactly given by the 1-jet of the function u.

Lemma 1.2.4. *Let u be a smooth function $X \to \mathbb{R}$. Define $Z = X \times \mathbb{R}$ and let P be the first order contact bundle of Z. Define S as the graph of u and $U = \mathcal{T}(S) \subset P$. Then in the local coordinates introduced before (see formula (1.2)) the manifold U is given by*

$$U = \{\, (x, y, \xi) \in P \mid y = u(x), \xi_j = \frac{\partial u}{\partial x_j}(x) \,\}.$$

This means that in local coordinates U is equal to the image of the 1-jet of the function u.

Proof. This follows almost immediately from the definition of the local coordinates for the contact bundle. Choose local coordinates $x_1, \ldots, x_n$ for X. Then $(x_1, \ldots, x_n, y)$ form local coordinates for Z. In the local coordinates the graph of u is, by definition,

$$U = \{\, (x, y) \in Z \mid y = u(x) \,\}.$$

The tangent space at the point $z = (x, y)$ of the graph U is determined by the first order derivatives $\frac{\partial u}{\partial x_j}(x)$. In fact, the tangent space of U at the point z is given by

$$T_z U = \{\, (X, Y) \in T_z Z \mid Y = \sum_j \frac{\partial u}{\partial x_j}(x) X^j \,\}.$$

This tangent space corresponds to the point $(x, y, \xi) = (x, u(x), \frac{\partial u}{\partial x}(x))$ in the contact bundle. The image $\mathcal{T}(U)$ of U in P is therefore precisely given by the 1-jet of u. □

Using the local coordinates for the first order contact bundle and the previous lemma we can make the following observations. Take $X = \mathbb{R}^n$. The 1-jet bundle of X is $J^1(X) = X \times \mathbb{R} \times \mathbb{R}^n$. The fibers are diffeomorphic to $\mathbb{R}^n$ and therefore not compact. The first order contact bundle P of $Z = X \times \mathbb{R}$ is diffeomorphic to $Z \times \mathbb{P}^1$. The standard coordinates $z = (x, y)$ for Z induce projective coordinates $(z, [\eta])$ for P. The n-dimensional linear subspace of T_z transversal to the vector ∂_y form an open subset of P and on this subset we have the local coordinates (x, y, ξ). These local coordinates are precisely the coordinates for the jet bundle $J^1(X)$. The 1-jet bundle of X is therefore an open subset of P. It is clear that this subset is dense, even fiberwise. The fibers of P are the compactifications of the fibers of the bundle $J^1(X) \to J^0(X) = X \times \mathbb{R}$. These remarks are valid for general manifolds X. The first order jet bundle $J^1(X)$ is open and dense in the first order contact manifold P of $Z = X \times \mathbb{R}$. The fibers of $P \to Z$ are compactifications of the fibers of $J^1(X) \to J^0(X)$.

Example 1.2.5 Take $X = \mathbb{R}$. The function $u(x) = x^2$ has graph S in $Z = X \times \mathbb{R}$ given by the set $\{ (x, y) \in \mathbb{R}^2 \mid y = x^2 \}$. The submanifold $U = \mathcal{T}(S)$ of the first order contact bundle is given in the projective coordinates (induced by the coordinates for Z) by the set $\{ (x, y, [\eta_1, \eta_2]) \in P \mid y = x^2, [\eta] = [-2x : 1] \}$, which we can identify with $\{ (x, y, \xi) \mid y = x^2, \xi = 2x \}$. Note that this U is an integral manifold of the contact structure C, because the contact structure is given by the form $\omega = dy - \xi dx$.

The manifold U given by the set $(x, x, [0 : 1])$ is a smooth submanifold of P, but *not* an integral manifold of the contact structure. This follows from the fact that $\{ (x, y) \mid x = y \}$ is a smooth manifold of Z, but the tangent space is given by $[1 : 1]$ and not $[0 : 1]$. The contact form restricted to U is $\omega|_U = (dy - \xi dx)|_U = 1 - 0 \cdot 1 = 1 \neq 0$. The contact form is nonzero, so U is not an integral manifold of the contact structure. $\oslash$

1.3 Obstructions

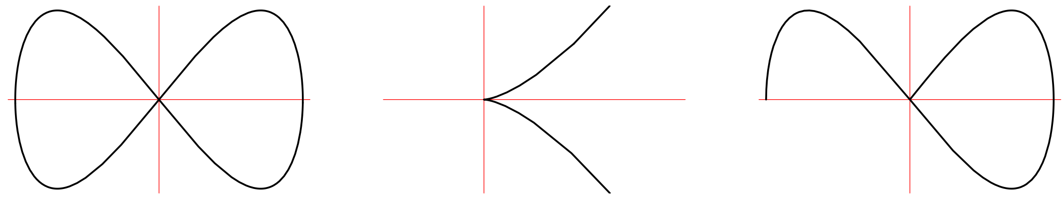
Let U be an integral manifold of the contact structure C. If the projection $\pi|_U$ is an embedding of U into Z, then $S = \pi_Z(U)$ is a smooth submanifold. Since π is smooth $\pi|_U$ can fail to be an embedding in only three ways: $\pi|_U$ is not injective, $T\pi|_{TU}$ is not injective or $(\pi|_U)^{-1}$ is not continuous. For each of the cases where the projection fails to be an embedding, we will give a typical example. The examples illustrate that these cases often correspond to the situation where the projection S is a submanifold with a certain type of singularity that is not present in the manifold U. In all cases we take $Z = \mathbb{R}^2$, so $P \cong Z \times \mathbb{P}^1$.

a) The standard coordinates x, y for $\mathbb{R}^2$ induce the coordinates $p = (x, y, [\eta])$ for P. We define the map $\phi : [0, 2\pi] \to P$ by $t \to (\sin t, \frac{\sin 2t}{2}, [\cos t : \cos 2t])$ (here π is not the projection but the real number $3.14\cdots$). It is clear that the image U of ϕ is a smooth submanifold of P. It is also an integral manifold of the contact structure, as can easily be checked.

   The projection of U to Z is equal to the figure eight in the plane (defined by the equations $x^2 = y^2 + x^4$). The figure eight is a smooth 1-dimensional submanifold except at the origin where it has a node-type singularity. The projection π^P_Z fails to be an embedding at the origin, because π^P_Z is not injective ($\pi^P_Z(\phi(0)) = \pi^P_Z(\phi(\pi))$, while $\phi(0) \neq \phi(\pi)$).

b) We consider the map $\psi : t \to (t^2, t^3, [2 : 3t])$. Again one can easily check that the image U of ψ is a smooth integral submanifold of C. The projection of U is equal to the image of the map $t \to (t^2, t^3)$. The image is the *cusp*, which is not smooth at the origin where it is continuous but not differentiable.

c) The last case is the situation where both π and $T\pi$ are injective when restricted to U, but the projection is not an embedding because the inverse mapping $\pi^{-1}|_S$ is not continuous. We can take for example, $s(t) = \frac{3\pi}{2} e^{-t}$ and $t \to (\sin(s(t)), \sin(2s(t))/2)$ for $0 < t < \infty$.



(a) The projection is an immersion, but not injective.

(b) The projection is not an immersion.

(c) The manifold is not compact, the inverse mapping is not continuous.

Figure 1.3: Examples for the three cases where the projection of a smooth integral submanifold of P is not a smooth submanifold of Z.

1.4 Partial differential equations on contact bundles

We saw in section 1.1 that a first order partial differential equation is defined by a function on the first order jet bundle. Since the 1-jet bundle J^1X is embedded in the first order contact bundle, it is natural to try to extend the partial differential equation to this contact bundle. We have also notices that a partial differential equation is unchanged if the defining function is multiplied by a nonzero function. These observations lead to the following definition.

Definition 1.4.1. Let (P, C) be the first order contact bundle of a manifold Z. A first order partial differential equation is a smooth hypersurface M that is transversal to the fibers of P. An n-dimensional integral manifold U of P is called a solution of the partial differential equation if U is contained in M.

What is the relation between this definition of a partial differential equation on the first order contact bundle and the classical partial differential equation as an equation on the jet bundle?

Suppose we have a normal first order partial differential equation $f = 0$ defined by a smooth function f on an open subset $\Omega \subset J^1(\mathbb{R}^n)$. Since $J^1(X)$ is embedded in the first order contact bundle P of $Z = \mathbb{R}^n \times \mathbb{R}$, the open subset Ω is also an open subset of P. A very natural condition on the partial differential equation $f = 0$ is to assume that $\frac{\partial f(x,y,\xi)}{\partial \xi} \neq 0$. This condition implies that we can locally solve the equation for one of the partial derivatives and it also guarantees that the equation has locally unique solutions. If we assume $\frac{\partial f}{\partial \xi}$ is surjective, then $M = \ker f$ is a smooth hypersurface in P that is transversal to the projection $P \to Z$. From the partial differential equation $f = 0$ we have constructed a partial differential equation on the contact bundle in the sense of the definition above.

Every solution U of the generalized partial differential equation is by definition an integral manifold of the contact structure C and is contained in M. If the projection of U to Z can be written as the graph of a smooth function u in X, then it follows from lemma (1.2.4) and the definition of M that u satisfies the partial differential equation. So every generalized solution gives rise to a candidate solution for the classical partial differential equation by projection to Z and then trying to write this projection as the graph of a function. This candidate solution does not always give a smooth solution to the partial differential equation, because of the obstructions mentioned in section 1.3 and the fact that the projection of U in Z can happen to be non-transversal to the fibers of the projection $Z \to X$. The other way around there are no problems with obstructions or non-transversality. Every solution u to the partial differential equation defines an integral manifold in P by taking $U = \mathcal{T}(S)$, where S is the graph of u. From the definitions and theorems in section 1.2 it follows that the manifold U is contained in M, so U is a generalized solution to the partial differential equation.

Under some minor conditions (transversality, no obstructions when projecting) there is one-to-one correspondence between solutions of the partial differential equation $f = 0$ and the solutions of the partial differential equation M on the first order contact bundle.

Example 1.4.2 We consider the simple case $X = \mathbb{R}$ and $Z = X \times \mathbb{R} = \mathbb{R}^2$. In this case a partial differential equation is always an ordinary differential equation that describes a relation between the function and its derivatives. Consider the differential equation on the 1-jet bundle $J^1(X)$ given by $f(x,u,u') = u' - c$, where c is an arbitrary constant. We can extend this equation to a differential equation on the first order contact bundle $P \cong Z \times \mathbb{P}^1$. The extension is given up to a factor by a function on P that has the same zero set as f on $J^1(X) \subset P$. We can take for example,

$$\tilde{f} : Z \times \mathbb{P}^1 \to \mathbb{R} : (x, u, [\eta_1 : \eta_2]) \mapsto \frac{(\eta_1 - c\eta_2)^2}{\eta_1^2 + \eta_2^2}.$$

On $J^1(X)$ we have $\eta_2 \neq 0$ and $\xi = -\xi_1/\xi_2$ corresponds to u'. Note that $\tilde{f}$ restricted to $J^1(X)$ is not equal to f, but their zero sets are equal on $J^1(X)$ so they define to same differential equation on $J^1(X)$. $\oslash$

Chapter 2

Higher order contact bundles and contact transformations

In the previous section we defined the first order contact bundle as a fiberwise compactification of the first order jet-space $J^1(X)$. We also introduced the first order contact structure and integral manifolds of the first order contact bundle. Using these structures we were able to define a first order partial differential equation as a smooth submanifold of the contact bundle. The next step is to generalize this construction and allow partial differential equations of arbitrary order to be formulated independent of a choice of coordinates.

In the following sections we will define the second order contact bundle and discuss some of its properties. Just like the first order contact bundle describes the 'first order derivative' of a submanifold, the second order contact bundle describes the second order derivatives. After that we will indicate how we can generalize the concept of a contact bundle to arbitrary order. In the second part of this chapter we will discuss contact transformations and define Legendre bundles.

2.1 The second order contact bundle

Let (P, C) be a first order contact bundle. We recall the definition of the Lie brackets modulo the subbundle in the appendix (lemma A.2.3). The Lie brackets modulo the subbundle define an anti-symmetric bilinear form on the contact structure $C \subset TP$. The contact structure C is given by a contact form ω (which is a 1-form up to a nonzero factor). In suitable local coordinates this contact form can be chosen as $\omega = dy - \xi^T dx$.

Lemma 2.1.1. *Let (P, C) be a first order contact bundle and p a point in P. The Lie brackets modulo the subbundle define a non-degenerate symplectic form on the contact structure C_p.*

Proof. It is clear that the Lie brackets modulo the subbundle are bilinear and anti-symmetric. Therefore we only need to check that the mapping $C_p \times C_p \to T_pP/C_P$ given by $(X_p, Y_p) \mapsto [X_p, Y_p]_{/C_p}$ is non-degenerate. We will reformulate the non-degeneracy of the Lie brackets modulo the subbundle in terms of the contact form ω for the contact structure C. For a differential form ω and vector fields X, Y we have the following identity (see [15, p. 135])

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]).$$

Since ω_p is zero when restricted to C_p, the equation reduces to

$$(d\omega)_p(X_p, Y_p) = -\omega_p([X_p, Y_p]_{/C_p}).$$

for vectors X_p, Y_p in C_p. The space T_pP/C_p is one-dimensional and ω_p is nonzero on T_pP/C_p so the element $[X_p, Y_p]_{/C_p}$ is uniquely determined by the left hand side of the last formula. The Lie brackets modulo the subbundle are therefore non-degenerate if and only if the differential $(d\omega)_p$ is non-degenerate on the contact structure C_p.

In local coordinates we have $d\omega = \sum_j dx^j \wedge d\xi^j$. This form is non-degenerate on the (x, ξ)-space. The contact structure C_p is the tangent space of the (x, y, ξ) space under the extra condition that $\omega = dy - \xi dx = 0$. This extra condition determines Y uniquely for any element (X, Y, Ξ) in C_p once X and Ξ are known. This means that the form $d\omega_p$ is non-degenerate on C_p. □

Remark 2.1.2 In the literature this lemma is often stated in terms of the Cauchy space of a distribution.

Definition 2.1.3. Let (M, E) be a vector subbundle. A vector $X \in E_m$ is called *Cauchy characteristic* if for all $Y \in E_m$ we have $[X, Y]_{/E_m} = 0$. The set of Cauchy vectors is called the *Cauchy characteristic space* of E at m and is denoted by $\mathrm{C}(E)_m$.

The conclusion of the previous lemma in terms of the Cauchy space is that for a contact bundle (P, C) the Cauchy space $\mathrm{C}(C)$ is equal to zero. ⊘

The important point of the previous paragraphs is that on the contact structure C we have a natural symplectic form (defined up to a nonzero factor either by the Lie brackets modulo the subbundle or the exterior derivative of the contact form). This symplectic structure is needed to define the second order contact bundle. In a bit informal way we can say that the second order contact bundle is defined by adding to each point of the first order contact bundle all possible tangent spaces of n-dimensional integral manifolds.

Definition 2.1.4. Let (P, C) be the first order contact bundle for a smooth manifold Z. For $p \in P$ we define Q_p to be the space of Lagrange planes in C_p with respect to the symplectic form induced by the Lie brackets modulo the subbundle. The Q_p are smooth manifolds which form a smooth fiber bundle Q over P. The second order contact structure is the vector subbundle D of TQ defined by taking for each $L \in Q$ the fiber D_L equal to $D_L = (T_L\pi^Q_P)^{-1}(L)$.

The bundle Q together with the contact structure D is called the *second order contact bundle*. The projection $Q \to P$ is denoted by π^Q_P.

An equivalent definition of Q_p is to take Q_p equal to the set of all n-dimensional linear subspaces L of C_p such that $X, Y \in L$ implies $[X, Y]_{/C_p} = 0$. This means that Q_p is equal to the set of n-dimensional integral elements in the first order contact structure C_p, i.e. $Q_p = \mathcal{I}_n(C)_p$.

We will now describe how we can choose local coordinates on Q using local coordinates on Z. Suppose L is an element of Q_p, so L is a Lagrange plane in C_p for some point $p \in P$. We can choose coordinates x, y for Z such that in the local coordinates (X, Y, Ξ) for TP induced from the coordinates (x, y, ξ) for P the Lagrange plane L is transversal to the Ξ-coordinates. The Lagrange plane L can then be written as

$$L = \{\, (X, Y, \Xi) \in C_p \mid Y = \sum_j \xi_j X_j, \Xi_i = \sum_j h_{ij} X_j \,\}, \tag{2.1}$$

for an $n \times n$-matrix h. The matrix h is uniquely determined by L and the choice of coordinates $z = (x, y)$. An n-dimensional linear subspace of C_p defined by $Y = \xi_j X_j, \Xi_i = \sum_j h_{ij} X_j$ is a Lagrange plane if for all points $v = (X, Y, \Xi)$ and $w = (X', Y', \Xi')$ in L we have $d\omega_p(v, w) = 0$. This means that

$$\begin{aligned} d\omega_p(((X, Y, \Xi), (X', Y', \Xi')) &= (\sum_i dx^i \wedge d\xi^i)(((X, Y, \Xi), (X', Y', \Xi')) \\ &= \sum_i X_i \Xi'_i - X'_i \Xi_i = \sum_{i,j} X_i h_{ij} X'_j - X'_i h_{ij} X_j \\ &= \sum_{i,j} X_i X'_j (h_{ij} - h_{ji}) = 0. \end{aligned}$$

$$
\begin{tikzcd}
Q \ni L \arrow[r, mapsto] \arrow[d] & D_L \subset T_L Q \arrow[d] \\
P \ni p \arrow[r, mapsto] \arrow[d] & C_p \subset T_p P \arrow[d] \\
Z \ni z \arrow[r, mapsto] & T_z Z
\end{tikzcd}
$$

Figure 2.1: The second order contact structure

Since X and X' can be chosen arbitrarily this is equivalent to $h_{ij} = h_{ji}$. We have proved that the fact that L is a Lagrange plane is equivalent to h being a symmetric matrix. The coordinates (x, p, h), with h a symmetric $n \times n$-matrix, form local coordinates for Q. We will see later that the matrix h has an interpretation as the second order derivative matrix (the Hessian), when we locally write a submanifold of Q as the 2-jet of a function $X \to \mathbb{R}$.

Just as in the first order case we can describe the contact structure D by a system of Pfaffian forms. Such a system is called a system of second order contact forms. In the local coordinates introduced above the standard contact system is given by

$$
\begin{aligned}
\omega_0 &= dy - \xi^T dx, \\
\omega_i &= d\xi_i - \sum_j h_{ij} dx_j, \qquad \text{for } 1 \leq i \leq n.
\end{aligned} \tag{2.2}
$$

2.2 Properties of the second order contact bundle

For a smooth submanifold U of P we define $\mathcal{T} : U \to \mathcal{T}(U) \subset Q$ by sending $p \in U$ to its tangent space.

Theorem 2.2.1. *Let (P, C) be a first order contact bundle and (Q, D) the corresponding second order contact bundle.*

i) Let U be a smooth integral submanifold of P. Then $V = \mathcal{T}(U)$ is a smooth integral manifold of (Q, D). The projection π_P^Q restricted to V is a diffeomorphism that is the inverse of $\mathcal{T}$ on U. If V is a smooth integral manifold of Q and the projection of V to P is an embedding, then $U = \pi_P^Q(V)$ is a smooth integral manifold of P.

ii) Let U be a smooth submanifold of P that is equal to the 1-jet of a smooth function u as in lemma 1.2.4. Then $\mathcal{T}(U)$ is a smooth submanifold of Q. In the local coordinates defined in equation (2.1) for the second order contact bundle, U is given by the 2-jet of u, so

$$
U = \{\, (x, y, \xi, h) \in Q \mid y = u(x), \xi = u'(x), h = u''(x) \,\}.
$$

Just as for a first order partial differential equation, we can define a second order partial differential equation as a smooth hypersurface in an open subset of Q. For this hypersurface we require that it is transversal to the fibers of the bundle $Q \to P$. If we write the hypersurface as the zero set of a smooth function f, this transversality condition is equivalent to $\frac{\partial f}{\partial h} \neq 0$.

The projection of a smooth integral manifold of (Q, D) to P or Z is not always a smooth manifold. The same obstructions as for the first order contact bundle apply (see section 1.3). The projection π_P^Q or π_Z^Q can fail to be an embedding when π is not injective, the tangent map $T\pi$ is not injective or the inverse mapping is not continuous. We will not give an example for every possibility, because there are a lot of possibilities and they are similar to the examples in section 1.3. In the next example we illustrate one type of obstruction that cannot occur in the projection from the first order contact bundle to the base manifold.

Example 2.2.2 The set $y^2 = x^5, x \geq 0$ is not a smooth submanifold of $Z = \mathbb{R}^2$, but it is the projection of a smooth integral submanifold V in Q. Both projections $S = \pi^Q_Z(V)$ and $U = \pi^Q_P(V)$ are not smooth manifolds, because the tangent mappings $T\pi^Q_Z$ and $T\pi^Q_P$ are not injective.

The first order contact bundle has local coordinates (x, y, ξ) on the open subset P_0 of P, where P_0 is the set of points $p = (x, y, [\eta])$, where $\eta_2 \neq 0$ (here $\xi = -\eta_1/\eta_2$). For the second order contact bundle Q we have local coordinates (x, y, ξ, h) on an open subset $Q_0 \subset Q$. Define $\psi : \mathbb{R} \to Q_0$ by $t \to (t^2, t^5, \frac{5}{2}t^2, \frac{15}{4}t)$. It is easy to check that ψ is an embedding of $\mathbb{R}$ in Q_0, so the image $U = \mathrm{im}(\psi)$ is a smooth 1-dimensional submanifold of Q_0 and thus of Q. We can check that V is an integral manifold of the contact structure D. In local coordinates this contact structure is given by the Pfaffian forms (2.2). We see that these Pfaffian forms are zero on (the tangent space of) V, so V is an integral manifold.

The projection to Z of V is equal to the image of $t \to (t^3, t^5)$. This image is smooth outside the origin and has a higher order cusp singularity at the origin. The projection of V to P has image $t \to (t^3, t^5, \frac{5}{3}t^2)$, the image is again not smooth at the point $(x, y, \xi) = (0, 0, 0)$. ⊘

2.3 Prolongation

We have defined the first and second order contact bundles as generalizations of the 1-jet and 2-jet bundle, respectively. For these bundles we have seen that smooth hypersurfaces that are transversal to the fibers represent partial differential equations of order one or two. The integral manifolds of maximal dimension of the contact bundles that are contained in such a hypersurface represent solutions of the corresponding partial differential equation. By projecting the integral submanifolds to the base space we find local solutions of the partial differential equation, perhaps with some sort of singularity as explained in section 1.3 on obstructions. A natural question is how we can generalize these contact bundles to higher order contact bundles so we can also represent higher order partial differential equations. Before giving the definition of a higher order contact bundle, we will take a closer look on the construction of the first and second order contact bundles. This will then be a model for the definition of the higher order contact bundles.

Suppose Z is a smooth $n+1$-dimensional manifold with corresponding first and second order contact bundle (P, C) and (Q, D). We can say that Z is the zeroth order contact bundle. From the zeroth order contact bundle Z we have obtained the first order contact bundle by adjoining to each point $z \in Z$ all possible tangent spaces of n-dimensional submanifolds of Z. In a similar way the second order contact bundle was constructed by adjoining to every point in the first order bundle all possible tangent spaces of n-dimensional *integral* manifolds. Using these definitions, we can map n-dimensional manifolds of Z to n-dimensional integral manifolds in the first and second order contact bundle using the prolongation map $\mathcal{T}$. The map $\mathcal{T}$ acts by adding to a point in an integral manifold U the same point together with the tangent space of U at that point.

The tangent bundle TZ is a vector subbundle of TZ. With this vector subbundle we can define the first order contact bundle as $P = \mathcal{I}_n(TZ)$, just as the second order contact bundle was defined as $Q = \mathcal{I}_n(C)$. This structure suggests that we can define higher order contact bundles by an inductive definition. A possible definition for the third order contact bundle $(Q^{(3)}, D^{(3)}$ would be $Q^{(3)} = \mathcal{I}_n(D)$ together with $(D^{(3)})_x = (T_x\pi^{Q^{(3)}}_Q)^{-1}(x)$, where the first x in the formula should be considered as a point in $Q^\vee$ and the second x as an n-dimensional linear subspace of $T_{\pi(x)}Q$. This definition can be generalized to arbitrary Pfaffian systems (M, E) (not necessarily contact bundles) if we take some care.

Definition 2.3.1. Let (M, E) be a Pfaffian system, with $\dim M = n+1$. We define the prolongation of the Pfaffian system (M, E) by $(M^\vee, E^\vee)$ where

$$M^\vee = \mathcal{I}_n(E)^{\mathrm{ord}}, \qquad (E^\vee)_I = (T_I\pi^{M^\vee}_M)^{-1}(I). \tag{2.3}$$

Here $I_n(E)^{\mathrm{ord}}$ is the set of ordinary integral elements of $I_n(E)$. The condition of ordinary elements is a technical condition to make sure that $M^\vee$ is a smooth manifold. For more details see [7, section 3.1, Definition 1.7 and 1.9] or [11, paragraph 1.7 and 1.9].

We take for the zeroth order contact bundle the Pfaffian system (Z, TZ). In the notation introduced above, the first order contact bundle (P, C) is the prolongation $(Z^\vee, (TZ)^\vee)$ of the zeroth order contact bundle. In the same way the second order contact bundle (Q, D) is the prolongation $(P^\vee, C^\vee)$ of the first order contact bundle. We state here that for the prolongations of the contact bundle we always have that $\mathcal{I}_n(D)^{\mathrm{ord}} = \mathcal{I}_n(D)$ (i.e. all integral elements in contact bundles are ordinary). The proof of this can be given using the theory on integral elements in [7, section 3.1]. We do not give the proof here, but note that because all contact bundles are locally equivalent to the standard contact bundle and the notion of an ordinary integral element is a local notion, it suffices to prove the statement for the standard contact bundle of $\mathbb{R}^n$. These observations make the following definition of higher order contact bundles possible.

Definition 2.3.2. We define the k-th order contact bundle $Q^{(k)}$ of an $n+1$-dimensional manifold Z as the k-th iterated prolongation of the Pfaffian system (Z, TZ). Every n-dimensional integral submanifold U of the k-th order contact bundle $Q^{(k)}$ of Z has a prolongation $U^\vee \subset Q^{(k+1)}$ defined by $U^\vee = \mathcal{T}(U)$. Here the map $\mathcal{T}$ is defined as $\mathcal{T} : U \to Q^{(k+1)} : u \mapsto (u, T_u U)$.

There are natural generalizations of theorem 1.2.3 and 2.2.1 for these prolongations, see [11, Lemma 1.16]. In particular, it follows that the prolongation $U^\vee$ is an n-dimensional integral manifold of $Q^\vee$.

2.4 Contact transformations

In the theory of ordinary and partial differential equations, coordinate transformations (or variable substitutions) are often used to simplify equations or show that certain classes of equations are equivalent to a standard form. In this section we will define *contact transformations*, which are a generalization of ordinary coordinate transformations. We will see in the examples that the class of contact transformations is larger than the class of coordinate transformations.

Definition 2.4.1. Let (P, C) and (P', C') be contact manifolds. A (local) *contact transformation* is a (local) diffeomorphism $\Phi : P \to P'$ that preserves the contact structure. If P and P' are the contact manifolds corresponding to the base manifolds Z and Z', then a (local) diffeomorphism $Z \to Z'$ will be called a *point transformation.*

The statement that a contact transformation Φ preserves the contact structure means that for all $p \in P$ we have $T_p\Phi(C_p) = C'_p$. If we describe the contact structures by a contact forms ω and ω', so $C_p = \ker \omega_p$ and $C'_{p'} = \ker \omega'_{p'}$, then $\Phi^*\omega'$ is equal to ω multiplied by a nonzero function on P.

Lemma 2.4.2. *Let (P, C) en (P', C') be the contact manifolds corresponding to the manifolds Z and Z', respectively. Every point transformation $Z \to Z'$ induces a contact transformation $P \to P'$ that preserves the fibers of the bundle $P \to Z$ and $P' \to Z'$. Conversely, every contact transformation preserving the fibers is induced in this way by a unique point transformation.*

Proof. Let $\Phi : Z \to Z'$ be a diffeomorphism between the manifolds Z and Z'. Let (P, C) and (P', C') be the first order contact bundles of Z and Z' respectively. The diffeomorphism Φ induces diffeomorphisms $TZ \to TZ'$ and $T^*Z \to T^*Z'$ in a natural way by taking $T_z\Phi$ and $((T_z\Phi)^*)^{-1}$. Because the diffeomorphism $\Phi : T^*Z \to T^*Z'$ maps the linear spaces $(T_zZ)^*$ linearly to $(T_zZ')^*$ this diffeomorphism induces a diffeomorphism $P = P(T^*Z) \to P' = P(T^*Z')$. We will write $T\Phi$ for the diffeomorphism $P \to P'$ induced in this way. We have to prove that this diffeomorphism preserves the contact structure.

Introduce local coordinates (x, y) in Z and corresponding coordinates (x, y, ξ) for P. We then define local coordinates (x', y') for Z' by $(x', y') = \Phi(x, y)$. The coordinates (x', y') induce local coordinates (x', y', ξ') on P' and it is easy to check that $(x', y', \xi') = T\Phi(x, y, \xi)$. The implies that every element $(X, Y, \Xi) \in C_p$ is mapped to the element $(X', Y', \Xi') = (X, Y, \Xi)$ in C'_p. In particular the kernel of the contact form on P is mapped to the kernel of the contact form on P' and the contact structure is preserved. The diffeomorphism $\Phi : Z \to Z'$ therefore induces a contact transformation of the contact bundles $\Phi : P \to P'$. Note that every contract transformation obtained in this way preserves the fibers of the bundle $P \to Z$, i.e. the fiber $(\pi_Z^P)^{-1}(z)$ is mapped to $(\pi_{Z'}^{P'})^{-1}(\Phi(z))$.

For the converse statement we note that if a contact transformation Ψ preserves the fibers, then it induces a natural diffeomorphism $\Phi : Z \to Z'$. We can define Φ for example by $\Phi = \pi_{Z'}^{P'} \circ \Psi \circ (\pi_Z^P)^{-1}$. We need to check that $\Psi = T\Phi$. This follows because in local coordinates $\Phi^*(\omega') = dy - (\xi')^T dx$. Because Φ is a contact transformation, we know that $\Phi^*(\omega') = \rho(dy - \xi^T dx)$. This implies that $\xi' = \xi$ and $T\Phi = \Psi$. So the contact transformations that preserve the fibers of the bundles are precisely the contact transformations induced by coordinate transformations. $\square$

Example 2.4.3 Take $X = \mathbb{R}$, $Z = \mathbb{R}^2$ with coordinates (x, y) and $P = P(T^*Z)$. We have seen in lemma 2.4.2 that coordinate transformations $Z \to Z$ induce contact transformations $P \to P$. For example the coordinate transformations $f(x, y) = (x, y)$ and $g(x, y) = (y, x)$ of Z induce the contact transformations

$$f : (x, y, [\eta]) \to (x, y, [\eta]), \quad g : (x, y, [\eta_1, \eta_2]) \to (y, x, [\eta_2, \eta_1]).$$

We will now construct a contact transformation that is *not* induced by a (local) coordinate transformation. Choose local coordinates (x, y, ξ) for P and suppose Φ is a local contact transformation defined on the set where we can use these local coordinates. So we can write the contact transformation as $(x', y', \xi') = \Phi(x, y, \xi)$.

Because the contact structure must be preserved by Φ the pull-back of $\omega' = dy' - \xi' dx'$ must be zero on C_p. We calculate

$$\Phi^*\omega' = \frac{\partial y'}{\partial x}dx + \frac{\partial y'}{\partial y}dy + \frac{\partial y'}{\partial \xi}d\xi - \xi'(\frac{\partial x'}{\partial x}dx + \frac{\partial x'}{\partial y}dy + \frac{\partial x'}{\partial \xi}d\xi).$$

Because $dy = \xi dx$ on C_p we can eliminate dy from the expression for $\Phi^*\omega'$ if we restrict to C_p. By eliminating dy we find

$$\Phi^*\omega'|_{C_p} = (\frac{\partial y'}{\partial x} + \xi\frac{\partial y'}{\partial y} - \xi\xi'\frac{\partial x'}{\partial x} - \xi'\frac{\partial x'}{\partial y})dx + (\frac{\partial y'}{\partial \xi} - \xi'\frac{\partial x'}{\partial \xi})d\xi = 0. \tag{2.4}$$

In local coordinates C_p is given by element of the form $\{ (X, Y, \Xi) \in T_pP \mid Y = \xi X \}$. Because X and Ξ can be freely chosen it follows from (2.4) that the following two equations must hold in order for the contact structure to be preserved.

$$\begin{aligned} \frac{\partial y'}{\partial x} + \xi\frac{\partial y'}{\partial y} - \xi'\frac{\partial x'}{\partial x} - \xi\xi'\frac{\partial x'}{\partial y} &= 0 \\ \frac{\partial y'}{\partial \xi} - \xi'\frac{\partial x'}{\partial \xi} &= 0 \end{aligned} \tag{2.5}$$

From these equations we can construct many contact transformations. Note that the two contact transformations f and g mentioned earlier satisfy these two equations.

Suppose we take $x' = \xi$, $y' = y - x\xi$ and $\xi' = -x$. It is easy to check that these relations satisfy (2.5) are therefore define a contact transformation. It is also clear that this contact transformation does not preserve the fibers (the base coordinates (x, y) and the fiber coordinate ξ are interchanged) and is therefore not induced by a point transformation. Note that the graph of a function $l : X \to \mathbb{R}$ is mapped by this contact transformation to the graph of the function $\hat{l}(x, \xi) = l(x) - x\xi$. The function $\hat{l}$ is exactly the Legendre transformation of the function l. So the Legendre transformation of functions is a special case of the contact transformations defined here. $\oslash$

We have defined contact transformations of the first order bundle. In a similar way we can define higher order contact transformations. In this section we will only consider second order contact transformations.

Definition 2.4.4. Let (Q, D) and (Q', D') be second order contact bundles. A second order contact transformation is a (local) diffeomorphism $Q \to Q'$ that preserves the second order contact structure D.

Just as every point transformation induces a first order contact transformation, every first order contact transformation induces in a natural way a second order contact transformation. If $\Psi : P \to P'$ is a first order contact transformation, then we can define a second order contact transformation $T\Psi : Q \to Q'$ by sending the Lagrange plane $L \in Q_p$ to the plane

$$L' = T_p\Psi(L). \tag{2.6}$$

The linear space L' is a subset of C'_p, because Ψ is a first order contact transformation. It is straightforward to check that the plane L' is a Lagrange plane in C'_p, so L' defines an element of Q'_p. This means that $T\Psi$ is a diffeomorphism $Q \to Q'$. Using an argument similar to that in the proof of lemma 2.4.2, we can prove that $T\Psi(D_L) = D'_{T\Psi(L)}$. The second order contact structure is preserved and $T\Psi$ is a second order contact transformation.

There are more first order contact transformations then point transformations, but the second order contact transformations are the same as the first order transformations.

Theorem 2.4.5. *Let Q and Q' be the second order contact bundles corresponding to the first order contact bundles P and P', respectively. Every second order contact transformation $Q \to Q'$ is induced from a first order contact transformation $P \to P'$.*

Proof. This theorem is attributed to Bäcklund [4]. Bäcklunds paper is difficult to read and one may question whether Bäcklunds arguments are correct (and complete). The results are correct, see [19] for a modern proof. □

Finally we want to describe how a contact transformation changes a partial differential equation. Because a contact transformation preserves the contact structure, a contact transformation maps integral manifolds to integral manifolds. Let Φ be a contact transformation $(Q, D) \to (Q', D')$. Under this contact transformation every smooth hypersurface M in Q is mapped to a smooth hypersurface $M' = \Phi(M)$ in Q'. Let U be an n-dimensional integral manifold of D that is at the same time contained in M. Then U is a generalized solution of the partial differential equation represented by the surface M. We define $U' = \Phi(U)$. This implies that U' is an n-dimensional integral manifold of D' and U' is contained in M'. This means that Φ maps generalized solutions of M to generalized solutions of M'.

We can also describe the surfaces M using functions. If locally M' can be written as $M' = \ker f'$, then $M = \ker f$ with $f = \Phi^* f'$. Suppose $U' = \Phi(U)$ and suppose we can locally write both U and U' as the 2-jet of a function, i.e. U is the image of j^2u and U' is the image of j^2u'. Then j^2u is a solution of the partial differential equation $f = 0$ if and only if j^2u' is a solution of the equation $f' = 0$. In this way we see that contact transformations also induce an equivalence between classical partial differential equations.

2.5 Second order contact transformations in local coordinates

Let (P, C), (P', C') be the first order contact bundles corresponding to the same base manifold $Z = X \times \mathbb{R}$, $X = \mathbb{R}^n$. Let (x, y, ξ) and (x', y', ξ') be the standard coordinates for P and P'. For the second order contact bundles Q and Q' we then have the local coordinates (p, h) and (p', h'), respectively. Let $\Phi : P \to P'$ be a first order contact transformation, so we can write $(x', y', \xi') = \Phi(x, y, \xi)$. The induced second order transformation $T\Phi$ maps the point (p, h) to some point (p', h') in $Q'_{p'}$. Because $C_p = \{ (X, Y, \Xi) \in T_pP \mid Y = \xi^T X \}$ we can identify C_p with the space (X, Ξ) by identifying (X, Ξ) with the element $(X, \xi^T X, \Xi) \in C_p$. In a similar way we can identify $C'_{p'}$ with (X', Ξ'). Because Φ is a first order contact transformation we know that $T_p\Phi$ maps C_p onto $C'_{\Phi(p)}$ linearly. With the identifications made for C_p and $C'_{p'}$ we can therefore write $T_p\Phi$ as

$$T_p\Phi : C_p \to C'_{\Phi(p)} : \begin{pmatrix} X \\ \Xi \end{pmatrix} \to \begin{pmatrix} X' \\ \Xi' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} X \\ \Xi \end{pmatrix} \tag{2.7}$$

for unique submatrices a, b, c, d. Note that these submatrices are $n \times n$-matrices that depend on the base point p. Because $T_p\Phi$ is induced from a first order contact transformation there are some restrictions on the allowed coefficients a, b, c, d.

Theorem 2.5.1. *Let $\Phi : (P, C) \to (P', C')$ be a first order contact transformation. After a choice of local coordinates for P and P' and the identification of C_p and $C'_{p'}$ with the (X, Ξ)-space and the (X', Ξ')-space, as before, we can write $T_p\Phi = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Then the submatrices a, b, c, d must satisfy*

$$a^T c = c^T a, \tag{2.8}$$

$$b^T d = d^T b, \tag{2.9}$$

$$d^T a - b^T c = \rho(p) I, \tag{2.10}$$

where $\rho(p)$ is a nonzero function on P.

Proof. We write ω and ω' for the standard contact forms on P and P'. Because Φ preserves the contact structure, we know that $\Phi^*\omega' = \rho\omega$ for a smooth function ρ that is nowhere zero. Taking the exterior derivative and using that $\omega|_{C_p} = 0$ we find that

$$(\Phi^* d\omega')_p(F, F') = d\omega'_{\Phi(p)}(T_p\Phi(F), T_p\Phi(F')) = \rho(p) d\omega_p(F, F')$$

for $F, F' \in C_p$. If we write $F = (X, \Xi)$, $F' = (X', \Xi')$ and use that in local coordinates $d\omega = \sum_i dx^i \wedge d\xi^i$, $d\omega' = \sum_i dx'^i \wedge d\xi'^i$ this is equivalent to

$$\begin{gathered} d\omega'_{\Phi(p)}((aX + b\Xi, cX + d\Xi), (aX' + b\Xi', cX' + d\Xi')) = \rho d\omega_p((X, \Xi), (X', \Xi')) \\ (aX + b\Xi)^T (cX' + d\Xi') - (cX + d\Xi)^T (aX' + b\Xi') = \rho(X^T \Xi' - \Xi^T X') \\ (X^T a^T + \Xi^T b^T)(cX' + d\Xi') - (X^T c^T + \Xi^T d^T)(aX' + b\Xi') = X^T \rho \Xi' - \Xi^T \rho X' \end{gathered}$$

and therefore

$$X^T(a^T d - c^T b + \rho)\Xi' + \Xi^T(b^T c - d^T a - \rho)X' + X^T(a^T c - c^T a)X' + \Xi(b^T d - d^T b)\Xi' = 0.$$

Because this equation holds for arbitrary X, Ξ, X' and Ξ' we see that the conditions (2.8), (2.9) and (2.10) must be satisfied. $\square$

Pointwise there is always a local contact transformation that realizes every possible set a, b, c, d (as long as the conditions of theorem 2.5.1 are satisfied). We can take as first order contact transformation the quadratic transformation

$$\begin{pmatrix} x \\ y \\ \xi \end{pmatrix} \to \begin{pmatrix} ax + b\xi \\ x^T c^T b\xi + \frac{1}{2} x^T c^T a x + \frac{1}{2} \xi^T d^T b \xi + (d^T a - b^T c) y \\ cx + d\xi \end{pmatrix}. \tag{2.11}$$

One can check that if the coefficients a, b, c, d satisfy the conditions of theorem (2.5.1) this transformation defines a first order contact transformation for which the action of the corresponding second order contact transformation on C_p is represented by $T_p\Phi = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

In formula 2.7 we have given the action of second order contact transformation in the contact space C_p. The contact transformation also induces an action on the Lagrange planes of C_p. Recall that we have introduced local coordinates for these Lagrange planes in section 2.1.

A Lagrange plane L described by the local coordinates (p, h) is given by $\{ (X, Y, \Xi) \in T_p P \mid Y = \xi^T X, \Xi = hX \}$, $p = (z, \xi)$. Under a contact transformation $\Psi : P \to P'$ the Lagrange plane L is mapped to the Lagrange plane

$$\begin{aligned} L' &= \{ (X', Y', \Xi') \in T_{p'} P' \mid X' = aX + b\Xi, \Xi' = cX + d\Xi, Y' = (\xi')^T X' \} \\ &= \{ ((a + bh)X, Y', (c + bh)X) \in T_{p'} P' \mid X \in \mathbb{R}^n, Y' = (\xi')^T (a + bh)X \} \end{aligned}$$

If $a + bh$ is invertible we can substitute $X \mapsto (a + bh)^{-1} X'$ and write L' as

$$\begin{aligned} L' &= \{ (X', Y', (c + bh)(a + bh)^{-1} X') \in T_{p'} P' \mid X' \in \mathbb{R}^n, Y' = (\xi')^T X' \} \\ &= \{ (X', Y', h'X') \in T_{p'} P' \mid X' \in \mathbb{R}^n, h' = (c + dh)(a + bh)^{-1}, Y' = (\xi')^T X' \} \end{aligned}$$

This means that the action of the contact transformation on the Lagrange planes is given by $h \mapsto h' = (c + dh)(a + bh)^{-1}$. When calculating the action of a contact transformation in local coordinates we have to be careful because it can happen that $\det(a + bh) = 0$. However because not both $\det a$ and $\det b$ can be zero, the set of matrices h for which $\det(a + bh) = 0$ is always a closed, lower dimensional, subset of all matrices h. This means that in practice the condition that $\det(a + bh)$ must be nonzero is not a real problem.

For a contact transformation Ψ we can write in local coordinates $\tilde{p} = \Psi(p)$. Under this contact transformation a partial differential equation $\tilde{f}(\tilde{p}, \tilde{h}) = 0$ is transformed to

$$f(p, h) = \tilde{f}(\Psi(p), (c + dh)(a + bh)^{-1}) = 0. \tag{2.12}$$

Example 2.5.2 Let X be a smooth manifold and Φ a (local) coordinate transformation of X. If we choose local coordinates $x_1, \ldots, x_n$ for X and local coordinates (x, y) for $Z = X \times \mathbb{R}$ then we have local coordinates (x, y, ξ) for the first order contact bundle P of Z. A point $p = (x, y, \xi)$ in P is an n-dimensional subspace of $T_{(x,y)} Z$ which in local coordinates is given by

$$\{ (X, Y) \in T_{(x,y)} Z \mid Y = \xi^T X \}.$$

If we write the coordinate transformation Φ as $x' = \Phi(x)$, then the induced first order contact transformation is given by $(x, y, \xi) \mapsto (x', y', \xi')$ where $x' = \Phi(x)$, $y' = y$. We can calculate ξ' because we know that $Y' = (\xi')^T X'$, $Y' = Y = \xi^T X$ and $X' = (T_{(x,y)} \Phi) X$. For convenience we write $a = T_{(x,y)} \Phi$. It follows that $(\xi')^T a X = \xi^T X$ and therefore $X^T \xi = X^T a^T \xi'$. Because this is true for arbitrary X we conclude that $\xi' = (a^{-1})^T \xi$. The second order contact transformation induced by Φ then has a representation of the form (2.7) given by

$$\begin{pmatrix} a & 0 \\ 0 & (a^{-1})^T \end{pmatrix}.$$

In particular we see that a contact transformation induced from a point transformation is fiber preserving. The x and ξ coordinates are transformed separately and are not dependent on each other. $\oslash$

2.6 Abstract contact manifolds

Many properties of the first order contact bundle of a manifold can be generalized to an abstract contact manifold. This abstract contact manifold has many of the properties of the first order contact manifold of a smooth $n+1$-dimensional manifold Z as introduced in definition 1.2.1, but it does not have a natural projection to a base space. This generalization makes it easier to present some of the results where the base space is irrelevant and makes it easier to make references to the literature where the abstract contact manifold is used. In this thesis these more abstract definitions will only be used in an application of the theory in section 4.2.

Definition 2.6.1. Let M be a smooth manifold and E a codimension one vector subbundle of TM. The pair (M, E) is called an *abstract contact manifold* if one of the following equivalent conditions is satisfied

i) If we write $E = \ker\alpha$ for a (locally defined) 1-form α then $d\alpha$ is non-degenerate on every hyperplane where $\alpha = 0$.

ii) The contact structure E has Cauchy characteristics equal to zero. This means that the Lie brackets modulo the subbundle define a non-degenerate antisymmetric bilinear mapping from $E_m \times E_m \to T_m M/E_m$.

On the linear space $M = \mathbb{R}^{2n+1}$ with local coordinates (x, y, ξ), $x, \xi \in \mathbb{R}^n, y \in \mathbb{R}$ we define the standard contact structure E by the 1-form $dy - \xi dx$. We call M with the standard contact structure the standard contact manifold. Note that this standard contact manifold is precisely the 1-jet bundle of $\mathbb{R}^n$ as a submanifold of the first order contact bundle of $\mathbb{R}^n$. For any manifold X the 1-jet bundle $J^1(X)$ is an abstract contact manifold of dimension $2n + 1$. From the theory in the beginning of section 2.1 it follows that every first order contact bundle $P(T^*Z)$ of a base manifold $Z = X \times \mathbb{R}$ is an abstract contact manifold. The jet bundle $J^1(X)$ with the contact structure induced from the first order contact bundle of X is another example of an abstract contact manifold.

In this thesis we work most of the time with contact manifolds induced from a base space Z (see definition 1.2.1). The extension to abstract contact manifolds is not a very big step because every abstract contact manifold of dimension $2n+1$ is locally isomorphic to the standard contact manifold. See [2, paragraph 20.1, Darboux's theorem on contact structure] for a proof and more details. It is not true that every abstract contact manifold is contact equivalent to a contact bundle induced from a base manifold.

2.6.1 Legendre fibrations

Definition 2.6.2. Let (P, C) be an abstract contact manifold of dimension $2n + 1$. An n-dimensional integral manifold of P is called a *Legendre submanifold*.

Note that n is the maximum dimension for any integral manifold of P because of the non-integrability of the contact structure.

Example 2.6.3

i) Let $M = J^1(\mathbb{R}^n)$ be the first order jet of $\mathbb{R}^n$ with coordinates (x, y, ξ). The contact form is given by $dy - \xi dx$. The manifold U defined by taking x and y constant is a Legendre submanifold of $J^1(\mathbb{R}^n)$. The tangent space of U at a point m is given by all vectors $(0, 0, \Xi)$ in $T_m M$. It is clear that the contact form is zero on this tangent space, so U is indeed an integral manifold.

   The manifolds with x and y constant are precisely the fibers of the projection $M \to \mathbb{R}^n$. In general the fibers of the projection $P(T^*Z) \to Z$ are Legendre submanifolds for any manifold Z.

ii) Let X be a manifold with first order contact bundle $P = P(T^*(X \times \mathbb{R}))$. The 1-jet of any function $X \to \mathbb{R}$ is a Legendre submanifold of P.

⊘

In the definition of an abstract contact manifold there is no reference to a base space. Locally we can always find a base space because the contact manifold is locally equivalent to the standard contact structure. An abstract contact manifold with a global base space is called a Legendre bundle. A more precise definition is

Definition 2.6.4. Let M be a contact manifold with contact structure E and $M \xrightarrow{\pi} B$ a fiber bundle with $\dim E = 2n+1, \dim B = n$. The bundle is called a *Legendre bundle* if the fibers $\pi^{-1}(b)$, $b \in B$ of the projection are Legendre submanifolds of M.

We have the following theorems for abstract contact manifolds and Legendre bundles.

Theorem 2.6.5 (Theorem 2.11 in [11])**.** *Let (P, C) be an abstract contact manifold with* $\dim P = 2n+1$. *If p is a point in P and L is a Lagrange plane in C_p (so L is an integral element of the contact structure) then there is an open neighborhood P_0 of P and a Legendre fibration $\varpi : P_0 \to \mathbb{R}^{n+1}$ such that $L = \ker T_p \varpi$.*

Theorem 2.6.6 (Theorem 2.12 in [11])**.** *Let (M, E) be an abstract contact manifold and let $\varpi : M \to B$ be a Legendre bundle. We write $(P = P(T^*B), C)$ for the first order contact bundle of B. We define the map $\Phi : M \to P$ by $m \in M \to T_m\varpi(E_m) \in G_n(T_{\varpi(m)}B)$.*

Then Φ is a local diffeomorphism. Moreover, Φ is the unique local contact transformation from (M, E) to (P, C) such that $\varpi = \pi^P_B \circ \Phi$.

The two theorems implicate that the notions of abstract contact manifold, Legendre bundle and first order contact bundle are locally equivalent.

2.6.2 Oriented contact manifolds

In the previous section we stated that locally every Legendre bundle $M \to B$ is equivalent to the first order contact bundle of the base space B and indicated a natural local diffeomorphism. A trivial example where the mapping Φ is locally a diffeomorphism but not globally is the following. Let P be the first order contact manifold of the manifold Z. We define the bundle M by adding to each point $z \in Z$ twice the fiber P_z of the bundle $P \to Z$, i.e. $M = P \oplus P$. Is this case the fibers of the bundle are not connected.

An example where the fibers are connected are *oriented contact bundles*. In section 1.2 we defined the first order contact bundle by adding to every base point all possible tangent spaces of codimension 1 submanifolds. We can also add all *oriented tangent spaces* to get the oriented contact bundle. The oriented contact bundle of an $n+1$-dimensional manifold Z is the bundle obtained adding to each point $z \in Z$ all oriented n-dimensional linear subspaces. We will work out this construction in the following example for $Z = \mathbb{R}^{n+1}$. The general case is very similar to this example.

Example 2.6.7 The first order contact bundle of $Z = \mathbb{R}^{n+1}$ is equal to $P = \mathbb{R}^{n+1} \times \mathbb{P}^n$. The coordinates $x = (x_1, \ldots, x_n, x_{n+1})$ induce coordinates $(x, [\eta])$ for the contact bundle. Here the element $(x, [\eta])$ represents the tangent space $(X_1, \ldots, X_{n+1})$ at the point x for which $\sum_{j=1}^{n+1} \eta_j X_j = 0$.

For every point $\eta = (\eta_1, \ldots, \eta_{n+1})$ we write $\{\eta\}$ for the equivalence class of η modulo $\mathbb{R}_{>0}$. In this way we have coordinates $\{\eta\}$ for the sphere S^n. At each point $x \in Z$ an n-dimensional oriented linear subspace H of the tangent space is represented by a vector $\eta \in \mathbb{R}^n$ by taking $H = \eta^\perp$. The orientation of H is represented by the direction of η. The oriented contact bundle of $\mathbb{R}^{n+1}$ is given by $\mathbb{R}^{n+1} \times S^n$ and on this oriented contact bundle we have coordinates $(x, \{\eta\})$. The point $(x, \{\eta\})$ represents the oriented linear subspace space of $T_x\mathbb{R}^{n+1}$ given by the set $\{\, (X_1, \ldots, X_{n+1}) \in T_x\mathbb{R}^n \mid \sum_j \eta_j X_j = 0 \,\}$ with oriented normal vector equal to η.

It is not difficult to check that on the oriented contact bundle $M = \mathbb{R}^{n+1} \times S^n$ we can define a contact structure by defining for each point $m = (x, \{\eta\})$ the linear subspace

$$E_m = (T_m \pi^M_Z)^{-1}(H),$$

where H is the subspace of $T_{\pi(m)}\mathbb{R}^{n+1}$ perpendicular to η. This contact structure makes (M, E) into an abstract contact manifold. The map Φ from theorem 2.6.6 defines a local diffeomorphism from the oriented contact bundle of $\mathbb{R}^{n+1}$ to the first order contact bundle of $\mathbb{R}^{n+1}$. In the coordinates introduced above this map is

$$\Phi : (x, \{\eta\}) \mapsto (x, [\eta]). \tag{2.13}$$

The mapping between the fibers of M and the first order contact bundle is a twofold covering from the n-sphere onto projective n-space. $\oslash$

In general we have that if $M \to Z$ is a Legendre bundle and $\Phi : M \to P(T^*Z)$ is the map from theorem 2.6.6 then Φ induces a covering from the fibers M_z onto $P_z = P((T_zZ)^*)$. Since $P((T_zZ)^*)$ is diffeomorphic to the projective space $\mathbb{P}^n$ and the fundamental group of $\mathbb{P}^n$ is equal to $\mathbb{Z}_2$ for $n \geq 2$, this covering is either the identify covering or the twofold universal covering. In this way the fundamental group of the fibers P_z gives a strong restriction on the possible Legendre bundles with base space Z.

Chapter 3

Monge-Ampère equations

In this section we will look at Monge-Ampère equations in the framework developed in the previous sections. We take $X = \mathbb{R}^2$ with coordinates x_1, x_2. The first order contact bundle has local coordinates $p = (x, y, \xi) = (x_1, x_2, y, \xi_1, \xi_2)$. The second order contact bundle has local coordinates (p, h) where h is a symmetric 2×2-matrix representing the second order derivative matrix.

A Monge-Ampère equation (in 2 variables) is a partial differential equation of type

$$j \det h + \operatorname{tr}(hk) + l = 0. \tag{3.1}$$

Here j and l are functions of p, k is a 2×2-matrix depending on p and h stands for the Hessian of a function u. We also demand that not both j and k are zero. In other words, we demand that the dependence on h is non-trivial. Written out in coordinates this means that

$$j(p)(u_{xx}u_{yy} - u_{xy}^2) + k_{11}(p)u_{xx} + (k_{21}(p) + k_{12}(p))u_{xy} + k_{22}(p)u_{yy} + l(p) = 0.$$

We call the functions j, k and l the coefficients of the Monge-Ampère equation. If $j = 0$ then the equation is quasi-linear, i.e. linear in the second order derivatives.

We will show how contact transformations act on the Monge-Ampère equation. In particular we will see that the class of Monge-Ampère equations is closed under contact transformations. For Monge-Ampère equations with constant coefficients the contact transformations allow us to reduce the Monge-Ampère equation to a special type of quasi-linear equation. In the next sections this reduction will allow us to pointwise classify every Monge-Ampère equation according to some of its geometrical properties.

3.1 Contact transformation to quasi-linear equations

In theorem (2.4.5) we noted that every second order contact transformation is induced from a first order contact transformation. Therefore we only need to consider all first order contact transformations and show how they transform the differential equation (3.1)

Theorem 3.1.1. *The Monge-Ampère equations are closed under second order contact transformations. Under a second order contact transformation for which the action on the first order contact structure is given by formula (2.7), a Monge-Ampère equation with coefficients $\tilde{j}$, $\tilde{k}$ and $\tilde{l}$ is transformed to a Monge-Ampère equation with coefficients*

$$\begin{aligned} j &= \tilde{j} \det d + \operatorname{tr}[db^{co}\tilde{k}] + \tilde{l} \det b, \\ l &= \tilde{j} \det c + \operatorname{tr}[ca^{co}\tilde{k}] + \tilde{l} \det a, \\ k &= c^{co}\tilde{k}^{co}b + a^{co}\tilde{k}d + \tilde{j}c^{co}d + \tilde{l}(a^{co}b). \end{aligned} \tag{3.2}$$

Proof. Under such a contact transformation a Lagrange plane L described in local coordinates by h is transformed to $\tilde{h} = (c + dh)(a + bh)^{-1}$ (see formula (2.12)). The general Monge-Ampère equation $\tilde{j} \det \tilde{h} + \operatorname{tr}(h\tilde{k}) + \tilde{l} = 0$ is therefore transformed to

$$\tilde{j} \det \left[(c + dh)(a + bh)^{-1}\right] + \operatorname{tr} \left[(c + dh)(a + bh)^{-1}\tilde{k}\right] + \tilde{l} = 0. \tag{3.3}$$

We will now rewrite this equation. We will use several common facts of traces and determinants. The most important ones are summarized in appendix C.2. First we multiply the equation by $\det(a + bh)$, the left-hand side of equation (3.3) becomes

$$\begin{aligned}
&\tilde{j} \det(c + dh) + \operatorname{tr} \left[(c + dh)(a + bh)^{\mathrm{co}}\tilde{k}\right] + \tilde{l} \det(a + bh) \\
&\quad = \tilde{j}(\det c + \det d \det h + \operatorname{tr}[c(dh)^{\mathrm{co}}]) + \operatorname{tr}[ca^{\mathrm{co}}\tilde{k} + dha^{\mathrm{co}}\tilde{k} + c(bh)^{\mathrm{co}}\tilde{k} + dh(bh)^{\mathrm{co}}\tilde{k}] \\
&\qquad + \tilde{l}(\det a + \det b \det h + \operatorname{tr} a(bh)^{\mathrm{co}}) \\
&\quad = \left[\tilde{j} \det d + \operatorname{tr}[db^{\mathrm{co}}\tilde{k}] + \tilde{l} \det b\right] \det h \\
&\qquad + \operatorname{tr}[c(bh)^{\mathrm{co}}\tilde{k}] + \operatorname{tr}[dha^{\mathrm{co}}\tilde{k}] + \tilde{j} \operatorname{tr}[c(dh)^{\mathrm{co}}] + \tilde{l} \operatorname{tr}[a(bh)^{\mathrm{co}}] \\
&\qquad\quad + \tilde{j} \det c + \operatorname{tr}[ca^{\mathrm{co}}\tilde{k}] + \tilde{l} \det a \\
&\quad = \left[\tilde{j} \det d + \operatorname{tr}[db^{\mathrm{co}}\tilde{k}] + \tilde{l} \det b\right] \det h \\
&\qquad + \operatorname{tr}[hc^{\mathrm{co}}\tilde{k}^{\mathrm{co}}b] + \operatorname{tr}[ha^{\mathrm{co}}\tilde{k}d] + \tilde{j} \operatorname{tr}[hc^{\mathrm{co}}d] + \tilde{l} \operatorname{tr}[ha^{\mathrm{co}}b] \\
&\qquad\quad + \left[\tilde{j} \det c + \operatorname{tr}[ca^{\mathrm{co}}\tilde{k}] + \tilde{l} \det a\right].
\end{aligned}$$

This last equation is again a Monge-Ampère equation of type $j \det h + \operatorname{tr}[hk] + l = 0$ with coefficients j, k and l given by formula 3.2. This proves that the class of Monge-Ampère equations is closed under contact transformations and that the coefficients transform according to formula (3.2). □

Due to the fact that we multiply by $\det(a + bh)$ some strange things may happen if we are not careful with the domain of the contact transformation. For a second order contact transformation for which the coefficients a, b, c, d are given as in 2.7, the domain is given by the set of (p, h) for which $\det(a + bh) \neq 0$. For a fixed p this means that the domain of the contact transformation is $\mathrm{Symm}^2(\mathbb{R}^2) \setminus N$, with $N = \{\, h \in \mathrm{Symm}^2(\mathbb{R}^2) \mid \det(a + bh) = 0 \,\}$. A partial differential equation $f = 0$ defines a surface $M = \ker f$ in the fiber of the second order contact bundle. In the generic situation the surfaces M and N intersect transversally. This means that the image of M under the contact transformation is again a closed surface and defines a partial differential equation. There can be a problem if M and N do not intersect transversally. For example if $M \subset N$, the surface M has no points in common with the domain of the contact transformation.

Example 3.1.2 Consider for example the equation $\det h = 0$. Typical solutions are the linear functions $u(x) = \alpha + \beta x_1 + \gamma x_2$. Under the local contact transformation with coefficients $a = d = 0, b = -c = I$ the graphs of these solutions are mapped to the submanifolds given by $\tilde{x} = (\beta, \gamma)$, $\tilde{y} = 0$ and $\tilde{\xi} = -(x, y)$. These manifolds are exactly the fibers of the projection $(\tilde{x}, \tilde{y}) \to \tilde{x}$ and therefore do not define solutions of the transformed equation.

In this case the contact transformation is $\tilde{h} = h^{-1}$. The domain of the contact transformation is the set of symmetric matrices h with $\det h \neq 0$. In particular the surface M of the partial differential equation is equal to the set N where the contact transformation is not defined. If we apply formula (3.2) without regarding this problem, we find the new coefficients $j = 0$, $k = 0$ and $l = 1$. This corresponds to the 'partial differential equation $1 = 0$', which is obviously not well-defined. ⊘

Remark 3.1.3 In the definition of a Monge-Ampère equation by the formula $\tilde{j} \det \tilde{h} + \operatorname{tr}[\tilde{h}\tilde{k}] + \tilde{l}$ there are two degrees of freedom in the choice of the coefficients $\tilde{j}$, $\tilde{k}$ and $\tilde{l}$. The first degree of freedom is multiplying all coefficients by a constant $\lambda \neq 0$. The transformed Monge-Ampère equation with coefficients $j = \lambda\tilde{j}$, $k = \lambda\tilde{k}$ and $l = \lambda\tilde{l}$ defines the same equation. The second degree of freedom is in the matrix $\tilde{k}$. For example the matrices

$$\tilde{k}_1 = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \quad \text{and} \quad \tilde{k}_2 = \begin{pmatrix} k_{11} & k_{12} + \lambda k_{21} \\ (1-\lambda)k_{21} & k_{22} \end{pmatrix}$$

both define the same equation but are different if both $k_{21} \neq 0$ and $\lambda \neq 0$. We could restrict ourselves to symmetric matrices $\tilde{k}$, but this is not very convenient because the contact transformations can map a symmetric $\tilde{k}$ to a non-symmetric k. We therefore allow arbitrary matrices $\tilde{k}$ and keep in mind that some matrices define equivalent partial differential equations. The choice of a specific matrix $\tilde{k}$ does not affect the partial differential equation after a contact transformation. This can be proved by looking at the transformed coefficients (3.2) and checking the effect of different choices of $\tilde{k}$ on the new coefficients.

$\oslash$

Suppose we want to find a contact transformation that makes the equation quasi-linear. This corresponds to finding a contact transformation, with suitable coefficients a, b, c, d that depend on p, such that $j = 0$ for all points p. The equation $j = 0$ involves derivatives of the contact transformation and is therefore a partial differential equation. This partial differential equation is difficult to solve in general. However, if we only want the make the equation quasi-linear in one point, then the equation $j = 0$ becomes an algebraic equation. For an algebraic equation it is much simpler to find solutions.

For this reason, we will consider Monge-Ampère equations only in one point in the remainder of this chapter. An equivalent viewpoint is to consider Monge-Ampère equations with constant coefficients. This means that j, k and l are constant and do not depend on the first order coordinates p. This restriction allows us the prove that all Monge-Ampère equations are contact equivalent in one point to quasi-linear equations.

In section 2.5 we have seen that there is a large class of local second order contact transformations. This class of local second order contact transformations can be parameterized by four 2×2-matrices satisfying the conditions in theorem 2.5.1. Because we are only considering the transformation in one point a contact transformation is completely described by the coefficients a, b, c, d in formula (2.7). We will use these transformations to show that every Monge-Ampère equation is pointwise contact equivalent to a quasi-linear equation. In the following sections we will find out even more about the pointwise structure of the Monge-Ampère equations. In section 4.2 we will consider the possibility of transforming a Monge-Ampère equation to a quasi-linear one not only in one point but on an open set.

Theorem 3.1.4. *Every Monge-Ampère equation is at one point contact equivalent to one of the following three normal forms*

$$\begin{aligned} h_{xx} + h_{yy} &= 0, && \text{(elliptic)} \\ h_{xx} - h_{yy} &= 0, && \text{(hyperbolic)} \\ h_{xx} &= 0. && \text{(parabolic)} \end{aligned}$$

The elliptic normal form is contact equivalent to $\det h - 1 = 0$*, the hyperbolic normal form is contact equivalent to* $\det h + 1 = 0$ *and the parabolic normal form is contact equivalent to* $\det h = 0$.

Proof. To transform every equation into one of the normal forms, we construct a sequence of contact transformations that will reduce an equation to one of the desired normal forms. Together with the transformations described in remark 3.1.3 these transformations will allow us to prove the theorem. All transformations that we need are summarized in table 3.1 and 3.2.

We want to transform a general Monge-Ampère equation to a quasi-linear one. The transformed equation is quasi-linear if and only if $j = 0$, so we want to find a contact transformation that makes $\tilde{j} \det d + \operatorname{tr}[db^{\mathrm{co}}\tilde{k}] + \tilde{l} \det b = 0$. We consider several different cases for the original equation which is defined by the coefficients $\tilde{j}$, $\tilde{k}$ and $\tilde{l}$. In each case we will give a sequence of contact transformations that makes the equation quasi-linear and reduces the equation to one of the three normal forms.

Parameter	**Action**
$\lambda \neq 0$	$j = \lambda \tilde{j}$ $k = \lambda \tilde{k}$ $l = \lambda \tilde{l}$
None	$j = \tilde{j}$ $k = (\tilde{k} + \tilde{k}^T)/2$ $l = \tilde{l}$

Figure 3.1: Transformations on the coefficients of Monge-Ampère equations that leave the equation invariant

Conditions	**Parameters**	**Action**	**Type**
	$a = I$ $d = \lambda I$, $\lambda \neq 0$ $b = c = 0$	$j = \lambda^2 \tilde{j}$ $k = \lambda \tilde{k}$ $l = \tilde{l}$	1
	$a = \alpha^{-1}$, $\det \alpha = 1$ $d = \alpha^T$, $\lambda \neq 0$ $b = c = 0$	$j = \tilde{j}$ $k = a^{\mathrm{co}} \tilde{k} d = \alpha \tilde{k} \alpha^T$ $l = \tilde{l}$	2
	$a = d = 0$ $b = -c = I$	$j = \tilde{l}$ $k = -\tilde{k}^{\mathrm{co}}$ $l = \tilde{j}$	3
$\tilde{j} = 0$ $\tilde{k}$ is diagonal	$b = c = 0$ $\nu, \mu \neq 0$ $a = \mathrm{diag}(\nu^{-1}, \mu^{-1})$ $d = \mathrm{diag}(\nu, \mu)$	$j = 0$ $k = \mathrm{diag}(\nu^2, \mu^2) \tilde{k}$ $l = \tilde{l} \nu^{-1} \mu^{-1}$	4
$\tilde{j} = 1$ $\tilde{l} = \pm 1, 0$	$a = d = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ $b = -c = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$	$j = 0$, $l = 0$ $k = \mathrm{diag}(1, -\tilde{l})$	5
	$a = d = I$ $b = 0$ c arbitrary symm.	$j = \tilde{j}$ $l = \tilde{l} + \tilde{j} \det c + \mathrm{tr}[c\tilde{k}]$ $k = \tilde{k} + \tilde{j} c^{\mathrm{co}}$	6
	$a = d = I$ $b = -c = I$	$j = \tilde{j} + \mathrm{tr}[\tilde{k}] + \tilde{l}$ $k = (\tilde{j} + \tilde{l}) I$ $l = \tilde{l} - \mathrm{tr}[\tilde{k}] + \tilde{j}$	7

Figure 3.2: Basic contact transformations

Quasi-linear case If $\tilde{j} = 0$ the equation is quasi-linear. Using the second transformation from table 3.1 we make $\tilde{k}$ symmetric. We know from linear algebra that we can diagonalize the symmetric matrix $\tilde{k}$ using an orthogonal matrix. Using a contact transformation of type 2 with α equal to this orthogonal matrix we can diagonalize α and keep $\tilde{j} = 0$. Now with a transformation of type 4 in combination with the first transformation of table 3.1 with $\lambda = \pm 1$, we reduce the matrix $\tilde{k}$ to either $\operatorname{diag}(1, 1)$, $\operatorname{diag}(1, -1)$ or $\operatorname{diag}(1, 0)$. Finally we transform $\tilde{l}$ to zero using a transformation of type 7. The equation is now in one of the standard forms.

Monge-Ampère case We assume $\tilde{j} \neq 0$. First we make $\tilde{k}$ equal to a multiple of the identity using a transformation of type 7. If after this transformation $\tilde{j}$ equals zero we are in the quasi-linear case again and use the methods described previously. If $\tilde{j} \neq 0$ we transform $\tilde{k}$ into zero using a transformation of type 6. Now that $\tilde{k} = 0$ we can use a transformation of type 1 with the first transformation in table 3.1 to make $\tilde{j} = 1$ and $\tilde{l} = 0, \pm 1$.

Equivalence of cases Finally we want to prove that the three forms $\det h + \tilde{l}$, with $\tilde{l} = 0, \pm 1$ are contact equivalent to the three normal forms. To prove this consider the action of the contact transformation of type 5. One can easily check that this transformation transforms the classes of equations into each other.

To complete the proof we need to check that all transformations we have used are well-defined. We leave it to the reader to check that this is indeed the case. □

Remark 3.1.5 We have not yet proved that the three normal forms cannot be contact equivalent to each other. We could do this explicitly by working out a general contact transformation for one of the normal forms and trying to write the transformed equation as one of the other normal forms. We will obtain the result by using the theory in the followings sections on invariance of the type of a Monge-Ampère equation under contact transformations. ⊘

Remark 3.1.6 Note that the theorem does not imply that a Monge-Ampère equation is contact equivalent to a quasi-linear equation in every point h of the fiber. For example for the Monge-Ampère equation $\det h = 0$ has a special point for $h = 0$. The equation at the point $h = 0$ cannot be transformed to a quasi-linear equation by any contact transformation. The equation $\det h$ has a singularity for $h = 0$, which is already clear from the fact that the total derivative of the function $f : \mathbb{R}^3 \to \mathbb{R} : h = \det h$ is equal to $(h_{yy}, h_{xx}, -2h_{xy})$. This total derivative is not surjective for $h = 0$ and therefore the zero set $\det h = 0$ is not a smooth manifold at $h = 0$. A more detailed description of this singularity and the corresponding equations will be given in sections 3.3.3 and 3.3.4. ⊘

3.2 The type of a second order partial differential equation

The existence, uniqueness and behavior of the solutions of partial differential equations depend on the order of the equation, the number of independent variables and the type of the equation. For example for elliptic partial differential operators there are strong regularity theorems. For hyperbolic operators there is an existence and uniqueness theorem for the analytic case (this is the Cauchy-Kowalenski theorem. In this section we will not discuss the various results on partial differential operators of different types. For an overview of these results see [18].

We will define the type of a linear partial differential operator and generalize this definition of type to an arbitrary differential equation. After that we give an equivalent definition in more geometrical terms. This definition will be coordinate independent and is therefore more suitable in the framework of contact bundles. We will show that both definitions are equivalent and that the type is invariant under contact transformations.

3.2.1 Linear partial differential operators

In this section we briefly review some of the main definitions of linear partial differential operators. A more detailed overview can be found in [18]. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ we write $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$. A linear partial differential operator is given in local coordinates by an expression

$$P = \sum_{|\alpha| \leq r} c_\alpha(x) \partial^\alpha. \tag{3.4}$$

Here the summation is over all multi-indices α with $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n \leq r$. The linear partial differential equation corresponding to this operator is given by $Pu = 0$. For a partial differential operator we define the total symbol $P(\xi)$ and principal symbol $p(\xi)$ as

$$P(\xi) = \sum_{|\alpha| \leq r} c_\alpha i^{|\alpha|} \xi^\alpha, \qquad p(\xi) = \sum_{|\alpha| = r} c_\alpha i^{|\alpha|} \xi^\alpha.$$

The type of a partial differential operator is determined by its principal symbol if the partial differential operator is of principal type. An operator is of principal type if it dominates all operators of lower order. If the operator is not principal the type can still depend on the lower order terms of the operator. See [18, book II, Def. 10.4.4 and Def. 10.4.11] for precise definitions. A vector $\xi \in \mathbb{R}^n$ is called a characteristic vector for the operator P if $p(\xi) = 0$. If ξ is characteristic then also the hyperplane $\xi^\perp$ is called characteristic. The operator is called **elliptic** if for all $\xi \in \mathbb{R}^n$, $\xi \neq 0$ we have $p(\xi) \neq 0$ and **strictly hyperbolic** with respect to the hyperplane $\xi^\perp$ if for all $\eta \neq 0$ with η not proportional to ξ (or equivalently, $\eta \in \xi^\perp$) the polynomial $p(\eta + \tau\xi)$ has r simple zeros. The elliptic and hyperbolic operators are always of principal type.

We are mainly interested in second order partial differential equations. In this case the principal symbol is a quadratic form in the variables $\xi_1, \ldots, \xi_n$. An operator is elliptic if the corresponding quadratic form is positive of negative definite. The other types also translate to properties of the eigenvalues of the quadratic form.

Example 3.2.1 We consider the special case of a linear second order equation in two independent variables x and y. Every equation of this type can be written as

$$pu_{xx} + qu_{xy} + ru_{yy} + au_x + bu_y + cu_x + du = 0.$$

The principal symbol of this equation is $p(\xi) = p\xi_x^2 + q\xi_x\xi_y + r\xi_y^2$. The type is determined by the properties of the quadratic form. We write the principal symbol as

$$p(\xi) = -(\xi_x, \xi_y) Q \begin{pmatrix} \xi_x \\ \xi_y \end{pmatrix}, \qquad Q = \begin{pmatrix} p & q/2 \\ q/2 & r \end{pmatrix}.$$

The type of the equation is determined by the sign of $\det Q$. The equation is elliptic for $\det Q > 0$, hyperbolic for $\det Q < 0$ and parabolic if $\det Q = 0$. Other types, such as the ultrahyperbolic case, cannot occur here. For the three types there are three standard forms. These are the classical equations

Laplace equation:	$u_{xx} + u_{yy} = 0$	(elliptic)
wave equation:	$u_{xx} - u_{yy} = 0$	(hyperbolic)
heat equation:	$u_{xx} - u_y = 0$	(parabolic)

$\oslash$

3.2.2 Linearization of partial differential equations

We want to generalize the definition of the type of a linear partial differential operator to a definition of the type of an arbitrary partial differential equation. We will do this by linearizing the partial differential equation in a point. In local coordinates this linearization is given by a linear partial differential operator. We define the type of the partial differential equation in that point to be the type of its linearization.

Definition 3.2.2. Let X be a smooth manifold and f a function on $J^r(X)$. We assume the zero set of f is a smooth hypersurface M. For every point $m \in M$ we define the linearization P_m of f at the point m to be the operator on $C^\infty(X)$ defined by

$$(P_m v)(x) = \frac{d}{d\epsilon}|_{\epsilon=0} f(j^r u_\epsilon)(x)$$

for a smooth function u_ϵ that satisfies $(j^r u_0)(x) = m$ and $\frac{d}{d\epsilon}|_{\epsilon=0} u_\epsilon = v$. Here $x = \pi_X^{J^r(X)}(m)$.

The linearization P_m depends on the point m, but not on the specific choice of u_ϵ used to calculate $P_m v$. In local coordinates the linearization is given by a linear partial differential operator. Of this partial differential operator we can take the principal symbol. This principal symbol is invariantly defined if we consider the principal symbol as a function on the cotangent space $(T_x X)^*$, $x = \pi(m)$. For details on this invariant definition we refer again to [18, book I, page 151]. For our purposes we only need to know that we can calculate everything in local coordinates and that the result (principal symbol, type etc.) is independent of the choice of these local coordinates.

Example 3.2.3 Consider a second order partial differential equation in two variables given by

$$f(x, u, p, h) = 0$$

with $x = (x_1, x_2)$, $p = (u_{x_1}, u_{x_2})$ and $h_{ij} = \frac{\partial f}{\partial x_i \partial x_j}$. The principal symbol of the linearization of this partial differential equation at a point $m \in \ker f$ is given by

$$p_m(\xi) = -\sum_{i,j} \frac{\partial f}{\partial h_{ij}}(m)\, \xi_i \xi_j. \tag{3.5}$$

$\oslash$

Example 3.2.4 Consider the third order non-linear Korteweg-de Vries equation $u_t + 6uu_x - u_{xxx} = 0$. In this equation we substitute $u \to u + \epsilon U$. Then up to order ϵ^2 we have

$$\begin{aligned} &u_t + \epsilon U_t + 6uu_x + 6\epsilon u U_x + 6\epsilon U u_x - u_{xxx} - \epsilon U_{xxx} \\ &\qquad = (u_t + 6uu_x - u_{xxx}) + \epsilon(U_t + 6u_x U + 6uU_x - U_{xxx}) = 0. \end{aligned}$$

The linearization of the Korteweg-de Vries equation is given by the first order terms in ϵ, therefore the linearization at the point $m = (x, t, u, u_x, u_t, u_x x, \ldots, u_{yyy})$ is $P_m = -\partial_x^3 + \partial_t + 6u\partial_x + 6u_x$. The linearization is neither elliptic or hyperbolic because the principal symbol $p(\xi) = -i\xi_x^3$ is degenerate (the linearization is not of principal type). $\oslash$

Of course the linearization of the partial differential equation $f = 0$ depends on the function f. However if f' is a function with the same zero set M, i.e. $f' = gf$ for a smooth non vanishing function g, then $P'_m = g(m)P_m$. This means that the linearization of a partial differential equation is up to a constant factor determined by the hypersurface defining the equation. Because the type of a linear partial differential operator is unchanged when the operator is multiplied by a nonzero constant the type of a partial differential equation is determined by its corresponding hypersurface in the jet bundle. We can now define the type of a partial differential equation defined by a hypersurface.

Definition 3.2.5. Let M be a smooth codimension one hypersurface in a jet bundle or contact bundle. We define the type of a point $m \in M$ to be the type of the linearization P_m of a smooth function f on the bundle, for which locally M is the zero set.

3.2.3 Type of a Monge-Ampère equation

In the previous sections we have defined the type of a partial differential equation. In this section we calculate the type of a Monge-Ampère equation. Because the Monge-Ampère equation is not quasi-linear (linear in the highest order derivatives) we expect that the linearization of a Monge-Ampère equation depends on the point where it is linearized.

We start by writing $u_\epsilon = u + \epsilon U$ and substituting this into the general Monge-Ampère equation (3.1). If we write this out and calculate the terms linear in ϵ we find (taking $k = \left(\begin{smallmatrix} p & q/2 \\ q/2 & r \end{smallmatrix}\right)$)

$$\begin{aligned}
\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} & [j \det h_\epsilon + \operatorname{tr}[h_\epsilon k] + l] \\
& = j(u_{yy}U_{xx} + u_{xx}U_{yy} - 2u_{xy}U_{xy}) + pU_{xx} + qU_{xy} + rU_{yy}.
\end{aligned}$$

The linearization of the Monge-Ampère equation is therefore given by

$$P_m = ju_{yy}\partial_x^2 + ju_{xx}\partial_y^2 - 2u_{xy}\partial_x\partial_y + p\partial_x^2 + q\partial_x\partial_y + r\partial_y^2$$

with $m = (x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy})$. The principal symbol we can write as

$$\begin{aligned}
p(\xi) &= (\xi_x, \xi_y) \begin{pmatrix} ju_{yy} + p & -ju_{xy} + q/2 \\ -ju_{xy} + q/2 & ju_{xx} + r \end{pmatrix} \begin{pmatrix} \xi_x \\ \xi_y \end{pmatrix} \\
&= \xi^T Q \xi.
\end{aligned}$$

The type is determined by the sign of the determinant of the matrix in the previous formula. This determinant is equal to

$$\begin{aligned}
\det Q &= (ju_{yy} + p)(ju_{xx} + r) - (-ju_{xy} + q/2)(-ju_{xy} + q/2) \\
&= j(j(u_{xx}u_{yy} - u_{xy}u_{xy}) + pu_{xx} + qu_{xy} + ru_{yy}) + pr - q^2/4.
\end{aligned}$$

We see that in principle the first term of this last equation depends on the point m where we linearize. However this term equals $-l$ for all points m satisfying the equation and therefore

$$\det Q = -jl + pr - q^2/4 = -jl + \det k. \tag{3.6}$$

The Monge-Ampère equation is elliptic for $-jl + \det k > 0$, hyperbolic for $-jl + \det k < 0$ and parabolic for $-jl + \det k = 0$.

3.2.4 Invariance of type in local coordinates

If we transform a general Monge-Ampère equation then the type of the new equation is determined by $-jt + \det k$. We will calculate $-jt + \det k$ in terms of the old $\tilde{j}, \tilde{k}, \tilde{l}$ and the transformation coefficients a, b, c, d. For convenience we write $\alpha = \det a$, $\beta = \det b$, $\gamma = \det c$ and $\delta = \det d$.

$$
\begin{aligned}
-jl + \det k &= -(\tilde{j} \det d + \operatorname{tr}[db^{\mathrm{co}}\tilde{k}] + \tilde{l} \det b)(\tilde{j} \det c + \operatorname{tr}[ca^{\mathrm{co}}\tilde{k}] + \tilde{l}\delta) \\
&\qquad + \det(c^{\mathrm{co}}\tilde{k}^{\mathrm{co}}b + a^{\mathrm{co}}\tilde{k}d + \tilde{j}c^{\mathrm{co}}d + \tilde{l}(a^{\mathrm{co}}b)) \\
&= -(\tilde{j} \det d + \operatorname{tr}[db^{\mathrm{co}}\tilde{k}] + \tilde{l} \det b)(\tilde{j} \det c + \operatorname{tr}[ca^{\mathrm{co}}\tilde{k}] + \tilde{l}\delta) \\
&\qquad + \det(c^{\mathrm{co}}\tilde{k}^{\mathrm{co}}b) + \det(a^{\mathrm{co}}\tilde{k}d) + \tilde{j}^2 \det(c^{\mathrm{co}}d) + \tilde{l}^2 \det(a^{\mathrm{co}}b)) \\
&\qquad\quad + \operatorname{tr}[c^{\mathrm{co}}\tilde{k}^{\mathrm{co}}b(a^{\mathrm{co}}\tilde{k}d)^{\mathrm{co}} + \tilde{j}c^{\mathrm{co}}\tilde{k}^{\mathrm{co}}b(c^{\mathrm{co}}d)^{\mathrm{co}} + \tilde{l}c^{\mathrm{co}}\tilde{k}^{\mathrm{co}}b(a^{\mathrm{co}}b)^{\mathrm{co}} \\
&\qquad\qquad + \tilde{j}a^{\mathrm{co}}\tilde{k}d(c^{\mathrm{co}}d)^{\mathrm{co}} + \tilde{l}a^{\mathrm{co}}\tilde{k}d(a^{\mathrm{co}}b)^{\mathrm{co}} + \tilde{j}\tilde{l}c^{\mathrm{co}}d(a^{\mathrm{co}}b)^{\mathrm{co}}] \\
&= -\Big[\tilde{j}\gamma\delta + \tilde{j}\delta \operatorname{tr}[ca^{\mathrm{co}}\tilde{k}] + \tilde{j}\tilde{l}\alpha\delta + \tilde{j}\gamma \operatorname{tr}[db^{\mathrm{co}}\tilde{k}] + \operatorname{tr}[db^{\mathrm{co}}\tilde{k}] \operatorname{tr}[ca^{\mathrm{co}}\tilde{k}] \\
&\qquad + \tilde{l}\alpha \operatorname{tr}[db^{\mathrm{co}}\tilde{k}] + \tilde{j}\tilde{l}\beta\gamma + \tilde{l}\beta \operatorname{tr}[ca^{\mathrm{co}}\tilde{k}] + \tilde{l}^2\alpha\beta\Big] \\
&\qquad + \Big[\beta\gamma \det(\tilde{k}) + \alpha\delta \det(\tilde{k}) + \tilde{j}^2\gamma\delta + \tilde{l}^2\alpha\beta\Big] \\
&\qquad\quad + \operatorname{tr}[c^{\mathrm{co}}\tilde{k}^{\mathrm{co}}d^{\mathrm{co}}\tilde{k}^{\mathrm{co}}a + \tilde{j}c^{\mathrm{co}}\tilde{k}^{\mathrm{co}}bd^{\mathrm{co}}c + \tilde{l}\beta c^{\mathrm{co}}\tilde{k}^{\mathrm{co}}a \\
&\qquad\qquad + \tilde{j}\delta a^{\mathrm{co}}\tilde{k}c + \tilde{l}a^{\mathrm{co}}\tilde{k}db^{\mathrm{co}}a + \tilde{j}\tilde{l}c^{\mathrm{co}}db^{\mathrm{co}}a] \\
&= -\Big[\tilde{j}\tilde{l}\alpha\delta + \operatorname{tr}[db^{\mathrm{co}}\tilde{k}] \operatorname{tr}[ca^{\mathrm{co}}\tilde{k}] + \tilde{j}\tilde{l}\beta\gamma\Big] \\
&\qquad + \Big[\beta\gamma \det(\tilde{k}) + \alpha\delta \det(\tilde{k})\Big] \\
&\qquad\quad + \operatorname{tr}[c^{\mathrm{co}}\tilde{k}^{\mathrm{co}}d^{\mathrm{co}}\tilde{k}^{\mathrm{co}}a + \tilde{j}\tilde{l}c^{\mathrm{co}}db^{\mathrm{co}}a] \\
&= (-\tilde{j}\tilde{l} + \det \tilde{k})(\alpha\delta + \beta\gamma) \\
&\qquad - \operatorname{tr}[db^{\mathrm{co}}\tilde{k}] \operatorname{tr}[ca^{\mathrm{co}}\tilde{k}] + \operatorname{tr}[c^{\mathrm{co}}\tilde{k}^{\mathrm{co}}d^{\mathrm{co}}\tilde{k}^{\mathrm{co}}a] + \tilde{j}\tilde{l} \operatorname{tr}[c^{\mathrm{co}}db^{\mathrm{co}}a] \\
&= (-\tilde{j}\tilde{l} + \det \tilde{k})(\alpha\delta + \beta\gamma) - \operatorname{tr}[bd^{\mathrm{co}}\tilde{k}^{\mathrm{co}}(ac^{\mathrm{co}}\tilde{k}^{\mathrm{co}})^{\mathrm{co}}] + \tilde{j}\tilde{l} \operatorname{tr}[c^{\mathrm{co}}db^{\mathrm{co}}a] \\
&= (-\tilde{j}\tilde{l} + \det \tilde{k})(\alpha\delta + \beta\gamma - \operatorname{tr}[bd^{\mathrm{co}}ca^{\mathrm{co}}])
\end{aligned}
$$

So far we have not used the fact that the matrices a, b, c, d define a contact transformation and therefore satisfy the conditions from theorem (2.5.1). From equation (2.8) it follows that ac^{co} is symmetric. This means that

$$
\begin{aligned}
\operatorname{tr}[bd^{\mathrm{co}}ca^{\mathrm{co}}] &= \operatorname{tr}[ac^{\mathrm{co}}db^{\mathrm{co}}] = \operatorname{tr}[(c^{\mathrm{co}})^T a^T db^{\mathrm{co}}] \\
&= \operatorname{tr}[(b^{\mathrm{co}})^T d^T ac^{\mathrm{co}}] = \operatorname{tr}[d^T ac^{\mathrm{co}}(b^T)^{\mathrm{co}}] \\
&= \operatorname{tr}[d^T a(b^T c)^{\mathrm{co}}].
\end{aligned}
$$

This allows us to rewrite the equation as

$$
\begin{aligned}
-jl + \det k &= (-\tilde{j}\tilde{l} + \det \tilde{k})(\alpha\delta + \beta\gamma - \operatorname{tr}[d^T a(b^T c)^{\mathrm{co}}]) \\
&= (-\tilde{j}\tilde{l} + \det \tilde{k})(\det(d^T a) + \det(b^T c) - \operatorname{tr}[d^T a(b^T c)^{\mathrm{co}}]) \\
&= (-\tilde{j}\tilde{l} + \det \tilde{k}) \det(d^T a - b^T c) = (-\tilde{j}\tilde{l} + \det \tilde{k})\rho^2.
\end{aligned}
$$

In the last step we used condition (2.10) for a contact transformation. We see that the type of the transformed equation is equal to the type of the original equation because ρ^2 is strictly positive.

From this type invariance it follows that the three normal forms of Monge-Ampère equations from theorem 3.1.4 cannot be contact equivalent. This implies that at each point there are exactly three equivalence classes of Monge-Ampère equations.

3.2.5 Geometric type of a partial differential equation

In the previous section we have seen that the type of a partial differential equation given by $f = 0$ depends only on the hypersurface $M = \ker f$. We will use this to give another definition of the type of a second order partial differential equation, one that is independent of a choice of coordinates. The previous definition was independent of the choice of coordinates, but we had to make a choice of coordinates to be able to write down the function f. Before we give this new definition, we will first give some motivation for this definition. After that, we show that the old and the new definition are equivalent.

The type of a partial differential equation depends only on the hypersurface M. Because the type of a general partial differential equation in a point $p \in P$ is defined by the principal symbol of its linearization, the type is already determined by the tangent space of $M_p = M \cap Q_p$. We can make this more precise in the following way. From the linearization we only need the principal symbol, that is the highest order part of the total symbol. For a second order partial differential equation the highest order part is given by the variables defining the fibers of the bundle $Q \to P$. This means that the type in a point p is already determined by the tangent space of $M \cap Q_p$. The hypersurface M is a codimension one manifold transversal to the fibers of $Q \to P$. This transversality is a quite natural condition on a hypersurface defining a partial differential equation and was already discussed in section 1.4. From the transversality condition it follows that $M_p = M \cap Q_p$ is a codimension 1 surface in the fiber Q_p. The tangent space $T_L M_p$ of M_p at a point $L = (p, h) \in Q_p$ is a codimension one linear subspace of $T_L Q_p$. The tangent space $T_L Q_p$ is canonically isomorphic to $\mathrm{Symm}^2(L)$ (see appendix B.3). and this implies that the tangent space $T_L M_p$ is defined by a nonzero form $\xi \in \mathrm{Symm}^2(L)^*$. In turn $\mathrm{Symm}^2(L)^*$ is canonically isomorphic to $\mathrm{Symm}^2(L^*)$ (see lemma C.1.2 in appendix C.1). In this way we see that the type of a second order partial differential equation in a point p is completely determined by a bilinear form on L^*. It will turn out that in local coordinates this bilinear form corresponds to the principal symbol of the corresponding partial differential equation.

Theorem 3.2.6. *Let M be a smooth hypersurface in the second order contact bundle (Q, D), representing a second order partial differential equation. For every point $m = (p, h) = L \in M$ the tangent space of $M \cap Q_p$ at the point L is equal to the kernel of a unique bilinear form γ on L^* up to a positive factor in a natural way by the canonical identifications in theorem C.1.2 and B.3.1.*

The number of positive, negative and zero eigenvalues of γ as a quadratic form on L^* correspond (up to a minus sign) to the eigenvalues of the principal symbol of M at the point m.

Proof. The construction of the bilinear form γ up to a positive factor was already sketched in the paragraphs before the theorem. To prove the statement on the eigenvalues of the bilinear form we calculate γ in local coordinates. We assume we have the usual local coordinates(p, h) for the second order contact bundle. If the hypersurface M is defined by the zero set of a smooth function f, then the tangent space of M_p is equal to

$$\{ H \in T_L Q_p \mid \left.\frac{d}{dt}\right|_{t=0} f(p, h + tH) \} = \{ H \in T_L Q_p \mid \sum_{i,j} \frac{\partial f}{\partial h_{ij}} H_{ij} = 0 \}. \tag{3.7}$$

The tangent space of M_p is therefore determined by the form τ on $\mathrm{Symm}^2(L)$ given by $H \mapsto \sum_{i,j} \frac{\partial f}{\partial h_{ij}} H_{ij}$ (or any nonzero multiple of τ). In the notation of appendix C.1 we have $\tau = \frac{\partial f}{\partial h_{ij}} \eta_{ij}$. From the discussion in section C.1 and especially formula (C.3) on the identification in local coordinates we find that the bilinear form γ on L^* is given by

$$\gamma = \gamma_{ij} dX^i dX^j, \qquad \gamma_{ij} = \frac{\partial f}{\partial h_{ij}}. \tag{3.8}$$

This bilinear form can be identified with the expression (3.5) of the linearization of a partial differential equation to prove the second statement of the theorem. □

Definition 3.2.7 (Geometric definition of type of PDE)**.** Let M be a smooth hypersurface in a second order contact bundle Q that is transversal to the fibers of Q. Let γ be the bilinear form from theorem 3.2.6 corresponding to a point $m \in M$.

The point m is *elliptic* if γ is positive or negative definite and *hyperbolic* if γ has $n-1$ positive eigenvalues and one negative eigenvalue or one positive eigenvalue and $n-1$ negative eigenvalues.

The previous theorem makes that this new definition of the type of a point is in correspondence with the old definition 3.2.5. The new definition is defined in terms of the contact structure and is therefore contact invariant. This contact invariance is easier to establish and much more general than the explicit calculation in section 3.2.4.

3.3 Surfaces in standard symplectic space

In section 2.2 we have seen that a second order partial differential equation defines a smooth surface in the second order contact bundle Q. We consider the intersection of this surface with Q_p, the fiber of the second order contact bundle.

For every point p in the first order contact bundle the contact space C_p is a symplectic vector space. The elements L of Q_p are the Lagrange planes of C_p. After a choice of coordinates (x, y, ξ) for the first order contact bundle the contact space C_p can be written as the collection $(X, Y, \Xi) \in T_pP$ for which $Y = \sum_j \xi_j X_j$. We identify C_p with the (X, Ξ)-space as in section (2.5). The symplectic form on C_p induced by the Lie brackets modulo the subbundle is then given by $\sigma((X, \Xi), (X', \Xi')) = X^T\Xi' - \Xi^T X'$. In this way we can identify C_p with the standard symplectic space. Because C_p is diffeomorphic to the standard symplectic space $E = \mathbb{R}^4$ we will work in this section only with E. The notation is simpler in this case and we can always translate the results back to C_p if we want. The definition and notation used here for the standard symplectic space (E, σ) is that of appendix B.2.

We will calculate the closure in $\Lambda(E, \sigma)$ of the surfaces defined by three special partial differential equations. These three special cases correspond exactly to the normal forms of the tree classic linear second order partial differential equations of example 3.2.1. At the end of this section we will combine the results for these special cases with the contact transformations defined before. This will lead to a geometric description of the hypersurfaces in Q_p defined by Monge-Ampère equations in two variables.

Remark 3.3.1 Although the structure of the surfaces defined by the Monge-Ampère equations is interesting by itself, we will see that for *every* partial differential equation in two variables these same structures will appear in the tangent space of the second order contact bundle (instead of in the second order contact bundle itself). We refer to section 4.2 for more details on this connection between the quasi-linear equations in the second order contact bundle and surfaces defined by more general equations in the tangent space. ⊘

3.3.1 Elliptic case

The most simple case is the case where we consider the hypersurface defined by the standard elliptic equation $u_{xx} + u_{yy} = 0$. The Lagrange planes in C_p are given in local coordinates by a symmetric matrices h. The equation for h is $\operatorname{tr} h = 0$. The surface defined by this equation is given by $V = \{\, v = \left(\begin{smallmatrix} a & b \\ b & -a \end{smallmatrix}\right) \mid a, b \in \mathbb{R} \,\} \subset \mathrm{Symm}^2(\mathbb{R}^2)$. Here the surface V in $\mathrm{Symm}^2(\mathbb{R}^2)$ should be identified with a surface in the space of Lagrange planes $\Lambda(E, \sigma)$ using the map (B.4). We identify the plane V with its corresponding surface W in $\Lambda^0(E, \sigma, M_1)$. The surface W is given by all Lagrange planes of the form

$$\{ \begin{pmatrix} v \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ \alpha \\ \beta \end{pmatrix} \mid \alpha, \beta \in \mathbb{R} \} = \{ \begin{pmatrix} a\alpha + b\beta \\ b\alpha - a\beta \\ \alpha \\ \beta \end{pmatrix} \mid \alpha, \beta \in \mathbb{R} \} \tag{3.9}$$

where a and b are the parameters describing the surface. To determine the closure of W in $\Lambda(E, \sigma)$ we calculate the image of W in all four standard coordinate patches and determine the closure of these images.

We first consider the chart $\Lambda^0(E,\sigma,M_2)$. This set equal to the set of planes of the form

$$\{ \begin{pmatrix} \alpha \\ \beta \\ w\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \end{pmatrix} \mid \alpha,\beta\in\mathbb{R} \} \tag{3.10}$$

where $w \in \mathrm{Symm}^2(\mathbb{R}^2)$. Note that if v is an invertible matrix, we can substitute $\left(\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right)$ by $v^{-1}\left(\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right)$, so the set of planes (3.9) with $\det v \neq 0$ can also be written as

$$\{ \begin{pmatrix} \alpha \\ \beta \\ v^{-1}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \end{pmatrix} \mid \alpha,\beta\in\mathbb{R} \} = \{ \begin{pmatrix} \alpha \\ \beta \\ \frac{1}{a^2+b^2}\begin{pmatrix}a\alpha+b\beta\\ b\alpha-a\beta\end{pmatrix} \end{pmatrix} \mid \alpha,\beta\in\mathbb{R} \} \tag{3.11}$$

These planes correspond to the matrices $\frac{1}{a^2+b^2}\left(\begin{smallmatrix}-a & -b\\ -b & a\end{smallmatrix}\right)$ in $\mathrm{Symm}^2(\mathbb{R}^2)$ in the coordinate patch $\Lambda^0(E,\sigma,M_2)$, see formula (B.5). Note that in the limit $a\to\infty$ or $b\to\infty$ the matrix

$$\frac{1}{a^2+b^2}\begin{pmatrix}-a & -b\\ -b & a\end{pmatrix} \to \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} \tag{3.12}$$

So the zero matrix is in the closure of the image of W in the coordinate patch $\Lambda(E,\sigma,M_2)$. Translating this back to $\Lambda(E,\sigma)$ we conclude that the plane M_1 is in the closure of W. In a similar way we can check the closure of W in the other two charts. In these charts we find that the closure of W contains no other points then the point M_1. The conclusion is that $\bar{W} = W \cup \{M_1\}$. The closure $\bar{W}$ is compact because it is closed and $\Lambda(E,\sigma)$ is compact. If $\bar{W}$ is a smooth surface it must be diffeomorphic to a sphere. This is in fact the case and we will give a diffeomorphism $S^2 \to W$.

The sphere S^2 can be covered by two coordinate patches (these coordinates are induced from the stereographic projection of the circle). We write S^2 as the set $\{ (\theta,\phi) \mid \theta\in[0,\pi], \phi\in[0,2\pi] \}$ and define

$$\begin{aligned}
&U_1 = \{ (\theta,\phi)\in S^2 \mid \theta\neq 0 \},\\
&U_2 = \{ (\theta,\phi)\in S^2 \mid \theta\neq \pi \},\\
&\kappa_1 : U_1 \to \mathbb{R}^2 : (\theta,\phi) \mapsto (x,y) = (\frac{\sin\theta}{1-\cos\theta}\cos\phi, \frac{\sin\theta}{1-\cos\theta}\sin\phi),\\
&\kappa_2 : U_2 \to \mathbb{R}^2 : (\theta,\phi) \mapsto (x',y') = (\frac{\sin\theta}{1+\cos\theta}\cos\phi, \frac{\sin\theta}{1+\cos\theta}\sin\phi).
\end{aligned}$$

The two charts (U_1,κ_1), (U_2,κ_2) cover S^2. The coordinate transformation $\kappa_2\circ\kappa_1^{-1}$ is given by

$$(x,y) \mapsto (x',y') = (\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}). \tag{3.13}$$

On the set $\kappa_1(U_1\cap U_2) = \{ (x,y)\in\mathbb{R}^2 \mid (x,y)\neq(0,0) \}$ this coordinate transformation is a smooth map.

We are now in the position to define the a diffeomorphism between the sphere S^2 and the closure of W in $\Lambda(E,\sigma)$. We define $f : S^2 \to \Lambda(E,\sigma)$ by

$$\begin{aligned}
&(\theta,\phi)\in U_1 \mapsto L_1(v), \quad v = \begin{pmatrix}x & y\\ y & -x\end{pmatrix},\ (x,y) = \kappa_1(\theta,\phi)\\
&(\theta,\phi)\in U_2 \mapsto L_2(v), \quad v = \begin{pmatrix}x' & y'\\ y' & -x'\end{pmatrix},\ (x',y') = \kappa_2(\theta,\phi).
\end{aligned} \tag{3.14}$$

See formulas (B.4) and (B.5) for the definition of $L_1(v)$ and $L_2(v)$). It is clear that f is a smooth injective mapping on U_1 and U_2. The map f is also surjective because $f(U_1) = W$ and $M_1 = f(\theta=\pi,\phi=0)$ is in the image of U_2. So we only need to check that f is well-defined on the overlap of U_1 and U_2. From the coordinate transformations (3.13) and (B.6) it is easy to see that f is indeed well-defined on the overlap $U_1\cap U_2$.

3.3.2 Hyperbolic case

The standard form of a hyperbolic equation is $u_{xx} - u_{yy} = 0$. However for our calculations it is more convenient to use the equation $u_{xy} = 0$. These two equations are contact equivalent so the corresponding surfaces can be transformed into each other by a symplectic transformation. We proceed just as in the previous case. We take $V = \{ v = \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) \mid a, b \in \mathbb{R} \} \subset \mathrm{Symm}^2(\mathbb{R}^2)$. The surface W in $\Lambda(E, \sigma)$ is given by planes of the form

$$\{ \begin{pmatrix} v \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ \alpha \\ \beta \end{pmatrix} \mid \alpha, \beta \in \mathbb{R} \} = \{ \begin{pmatrix} a\alpha \\ b\beta \\ \alpha \\ \beta \end{pmatrix} \mid \alpha, \beta \in \mathbb{R} \}. \tag{3.15}$$

In the coordinate patch $\Lambda^0(E, \sigma, M_2)$ the image of W is given by the collection of matrices $\left(\begin{smallmatrix} -a^{-1} & 0 \\ 0 & -b^{-1} \end{smallmatrix}\right)$ for which $a \neq 0$, $b \neq 0$. The closure contains the extra points $\left(\begin{smallmatrix} 0 & 0 \\ 0 & c \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} d & 0 \\ 0 & 0 \end{smallmatrix}\right)$ with $c, d \in \mathbb{R}$. These points correspond to the Lagrange planes

$$\{ \begin{pmatrix} \alpha \\ \beta \\ 0 \\ c\beta \end{pmatrix} \mid \alpha, \beta \in \mathbb{R} \}, \quad \{ \begin{pmatrix} \alpha \\ \beta \\ d\alpha \\ 0 \end{pmatrix} \mid \alpha, \beta \in \mathbb{R} \}.$$

The closure in the two other coordinate patches contains two other points. We will show this explicitly for the coordinate patch $\Lambda^0(E, \sigma, M_4)$. The other case (for M_3) is similar and is left to the reader. Note that if $a \neq 0$ we can make the substitution $(\alpha, \beta) = (a^{-1}A, B)$. The planes (3.15) for which $a \neq 0$ can then be written as

$$\{ \begin{pmatrix} A \\ bB \\ a^{-1}A \\ B \end{pmatrix} \mid A, B \in \mathbb{R} \}.$$

From this we see that the image of W in $\Lambda^0(E, \sigma, M_4)$ is given by the matrices $\left(\begin{smallmatrix} -a^{-1} & 0 \\ 0 & b \end{smallmatrix}\right)$. The closure consists of the matrices $\left(\begin{smallmatrix} 0 & 0 \\ 0 & b \end{smallmatrix}\right)$ which in turn correspond to the Lagrange planes

$$\{ \begin{pmatrix} A \\ bB \\ 0 \\ B \end{pmatrix} \mid A, B \in \mathbb{R} \}.$$

For $b \neq 0$ these Lagrange planes correspond to the planes already found in the chart for M_2. For $b = 0$ we find one extra point in $\Lambda(E, \sigma)$: the Lagrange plane M_4. In the chart $\Lambda^0(E, \sigma, M_3)$ we find that also M_3 is in the closure of W. We will prove that the closure of W is diffeomorphic to a torus. The torus T we can describe using four coordinate patches (U_j, κ_j).

$$\begin{aligned}
U_1 &= \{ (\theta, \phi) \mid \theta \neq 0, \phi \neq 0 \}, \quad \kappa_1 : (\theta, \phi) \mapsto \left(\frac{\sin\theta}{1 - \cos\theta}, \frac{\sin\phi}{1 - \cos\phi}\right) \\
U_2 &= \{ (\theta, \phi) \mid \theta \neq \pi, \phi \neq \pi \}, \quad \kappa_2 : (\theta, \phi) \mapsto \left(\frac{\sin\theta}{1 + \cos\theta}, \frac{\sin\phi}{1 + \cos\phi}\right) \\
U_3 &= \{ (\theta, \phi) \mid \theta \neq \pi, \phi \neq 0 \}, \quad \kappa_3 : (\theta, \phi) \mapsto \left(\frac{\sin\theta}{1 + \cos\theta}, \frac{\sin\phi}{1 - \cos\phi}\right) \\
U_4 &= \{ (\theta, \phi) \mid \theta \neq 0, \phi \neq \pi \}, \quad \kappa_4 : (\theta, \phi) \mapsto \left(\frac{\sin\theta}{1 - \cos\theta}, \frac{\sin\phi}{1 + \cos\phi}\right)
\end{aligned}$$

The map from T to $\bar{W}$ we define by

$$
\begin{aligned}
&(\theta,\phi) \in U_1 \mapsto L_1(v), \qquad v = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix},\ (x,y) = \kappa_1(\theta,\phi) \\
&(\theta,\phi) \in U_2 \mapsto L_2(v), \qquad v = \begin{pmatrix} -x' & 0 \\ 0 & -y' \end{pmatrix},\ (x',y') = \kappa_2(\theta,\phi) \\
&(\theta,\phi) \in U_3 \mapsto L_3(v), \qquad v = \begin{pmatrix} -y' & 0 \\ 0 & -x \end{pmatrix},\ (x,y') = \kappa_3(\theta,\phi) \\
&(\theta,\phi) \in U_4 \mapsto L_4(v), \qquad v = \begin{pmatrix} -x' & 0 \\ 0 & y \end{pmatrix}\ (x',y) = \kappa_4(\theta,\phi).
\end{aligned}
$$

Because the patches U_1, U_2, u_3 and U_4 cover the torus this defines a map from the torus in the space of Lagrange planes. One can check that this map is a well-defined diffeomorphism between the torus and the closure of W. Injectivity is clear, subjectivity follows from the analysis of the closure of W in the previous paragraphs. The fact that this map is well-defined follows from the transformation formulas (B.6) in appendix B for the special case $a = x, b = 0, c = y$.

3.3.3 Parabolic case

This last case is more difficult than the previous cases. The hypersurface defined here turns out to have a single singularity.

We start with the equation $u_{xx}u_{yy} - u_{xy}^2 = 0$. Note that this is a Monge-Ampère equation with $j = 1$, $k = 0$ and $l = 0$. In the first coordinate patch $\Lambda^0(E,\sigma,M_1)$ the corresponding surface is given by $V = \{\, v = \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right) \mid a,b,c \in \mathbb{R},\ \det v = 0 \,\} \subset \mathrm{Symm}^2(\mathbb{R}^2)$. In this coordinate patch the surface looks like a cone. Indeed, if we substitute $a = z+x$, $b = y$ and $c = z-x$ then the condition $\det v = ac - b^2 = 0$ can be written as $(z+x)(z-x) - y^2 = z^2 - x^2 - y^2 = 0$, so $x^2 + y^2 = z^2$. The surface is therefore smooth for $(x,y,z) \neq 0$ but has a singularity at the point $(x,y,z) = 0$. We claim that the closure of the surface V in $\Lambda(E,\sigma)$ is smooth outside this singularity point. The closure is a constricted torus, see figure 3.4(b).

To describe the surface in the other coordinate patches we will parameterize the surface just as we did in the previous two cases. Because the surface is singular, there is not one parameterization that can cover the whole surface V. We will therefore use the following two parameterizations

$$
1: \quad \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} A & AB \\ AB & AB^2 \end{pmatrix}, \qquad A, B \in \mathbb{R} \tag{3.16}
$$

$$
2: \quad \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} CD^2 & CD \\ CD & C \end{pmatrix}, \qquad C, D \in \mathbb{R} \tag{3.17}
$$

Note that for all values of A, B, C and D the corresponding matrices have determinant zero. We will now consider the two parameterizations separately. Because both parameterizations are dense in V, they are dense in the closure of V. We therefore only need to determine the closure of the first parameterization and check that this closure is smooth outside the origin.

Case 1 We consider the parameterization (3.16). We have three sets of parameters to describe the surface V: the parameters a,b,c and x,y,z (each with one extra condition) and the unrestricted parameters A,B. The relation between A,B and the other parameters is $A = a = z+x$, $B = a^{-1}b = (z+x)^{-1}y$ and $AB = b = y$. By taking this parameterization we leave out the points for which $a = z+x = 0$ but $c = z-x \neq 0$. These points correspond to the line in the surface where '$B = \infty$'. In figure 3.3(a) there is a picture of the surface in the (x,y,z)-coordinates. The line that cannot be described with the parameters A,B is marked bold.

Because the matrices in (3.16) all have determinant equal to zero the image in the coordinate patch $\Lambda^0(E,\sigma,M_2)$ is empty. In the coordinate patch $\Lambda^0(E,\sigma,M_4)$ the first parameterization of the surface is given by the matrices

$$\begin{pmatrix} -A^{-1} & B \\ B & 0 \end{pmatrix}.$$

We see that the matrices $\left(\begin{smallmatrix} 0 & B \\ B & 0 \end{smallmatrix}\right)$ are in the closure. A limit sequence to these matrices is given for example by taking the limit $(x, y, z) \to \infty$ while holding the ratio $[x : y : z]$ fixed. For the parameters A, B this means that B is held fixed while $A \to \infty$. So the closure in $\Lambda^0(E, \sigma, M_4)$ contains an extra real line of points. We closure of the image of our surface in the coordinate patch for M_4 is given by all matrices of the form $\left(\begin{smallmatrix} a & b \\ b & 0 \end{smallmatrix}\right)$. This is a smooth surface that described the equation $u_{yy} = 0$.

In the patch for M_3 the matrices are given by

$$\begin{pmatrix} -A^{-1}B^{-2} & B^{-1} \\ B^{-1} & 0 \end{pmatrix}.$$

The closure of these matrices consists of two lines which have one point in common. The first line consist of matrices $\left(\begin{smallmatrix} 0 & D \\ D & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & B^{-1} \\ B^{-1} & 0 \end{smallmatrix}\right)$ for $D \in \mathbb{R} \setminus \{0\}$. These are the matrices for which $(x, y, z) \to \infty$ with fixed ratio. This line overlaps for a large part with the line we found in the patch for M_3. The second line is the line of matrices $\left(\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right)$. These matrices can be found by taking the limit $A, B \to \infty$ in such a way that the ratio $-A^{-1}B^{-2} = \frac{-1}{z-x}$ converges to the constant E. This second line is the line of matrices with '$B = \infty$' that we left out when we chose our parameterization. The two lines we found have one point in common, this point corresponds to the matrix $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ in the coordinate patch M_3.

In the closure of V in the four coordinates patched we have found an extra projective line. This line corresponds to the 'line at infinity' for the cone in figure 3.3(a).

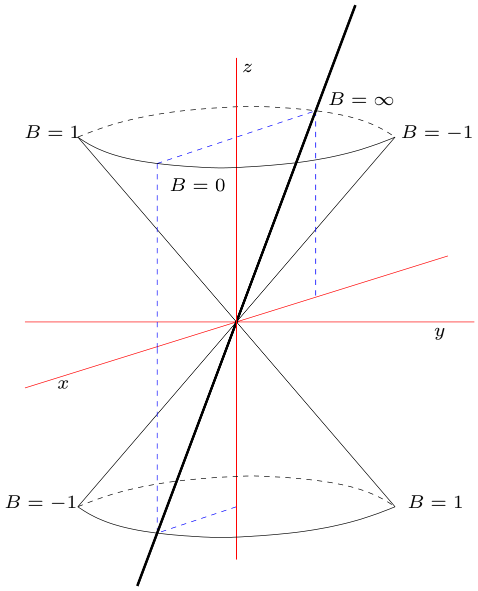
Case 2 It is not needed to work this case out as we have remarked before. To make the picture complete we give the main results. For the second parameterization we have $C = z - x$, $D = c^{-1}b = (z - x)^{-1}y$. The surface in the coordinate patch M_3 is given by the matrices $\left(\begin{smallmatrix} -C^{-1} & D \\ D & 0 \end{smallmatrix}\right)$. The closure is the line $\left(\begin{smallmatrix} 0 & D \\ D & 0 \end{smallmatrix}\right)$, $D \in \mathbb{R}$. In the coordinate patch for M_4 the surface looks like $\left(\begin{smallmatrix} -C^{-1}D^{-2} & D^{-1} \\ D^{-1} & 0 \end{smallmatrix}\right)$. The closure of this last surface contains the matrices $\left(\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right)$ (these are the matrices with $D = \infty$) and the matrices $\left(\begin{smallmatrix} 0 & D^{-1} \\ D^{-1} & 0 \end{smallmatrix}\right)$ (these are the matrices with $c = C = \infty$, these corresponds to the same Lagrange planes we found in the closure of the surface in the coordinate patch for M_3).

From the discussion above we see that the closure of our cone consists of identifying the 'points at infinity' of the cone with each other. These equations for the closure of the cone in the other coordinate patches define smooth surfaces. We can conclude that the closure of our surfaces is indeed a constricted torus.

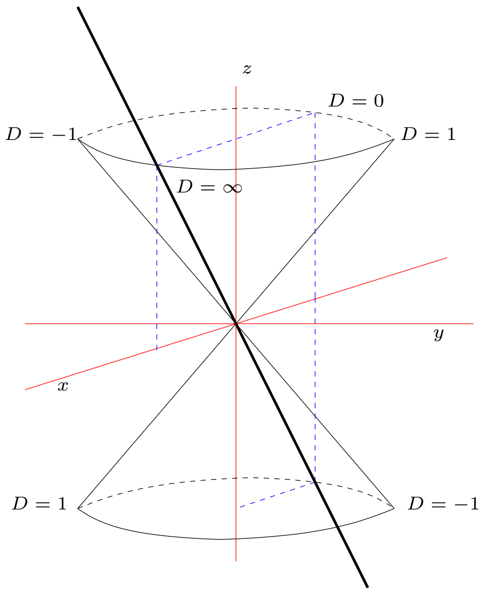
3.3.4 Geometric picture of surfaces in the Lagrange space

For three standard equations we have given a detailed description of the surface defined by these equations in the fiber of the second order contact bundle. Since all Monge-Ampère equations are contact equivalent to one of these equations and the contact transformations are diffeomorhpisms of the fibers of the bundle, every surface corresponding to a Monge-Ampère equation corresponds to a sphere, torus or constricted torus.

Consider the Monge-Ampère equation $\det h = \epsilon$, where ϵ is a parameter. For $\epsilon > 0$ the equation is elliptic, for $\epsilon = 0$ the equation is parabolic and for $\epsilon < 0$ the equation is hyperbolic. By varying the parameter ϵ we can therefore make a transition from an elliptic to a hyperbolic equation. The surfaces in the contact bundle that correspond to the equation make a corresponding transition. For $\epsilon < 0$ the surface is a torus. When $\epsilon \uparrow 0$ the torus becomes squeezed. For $\epsilon = 0$ the torus is deformed into a constricted torus. This is a smooth manifold that has one cone type singularity (see figure 3.4(b)). For $\epsilon > 0$ the singularity disappears and the surfaces has become diffeomorphic to a sphere.

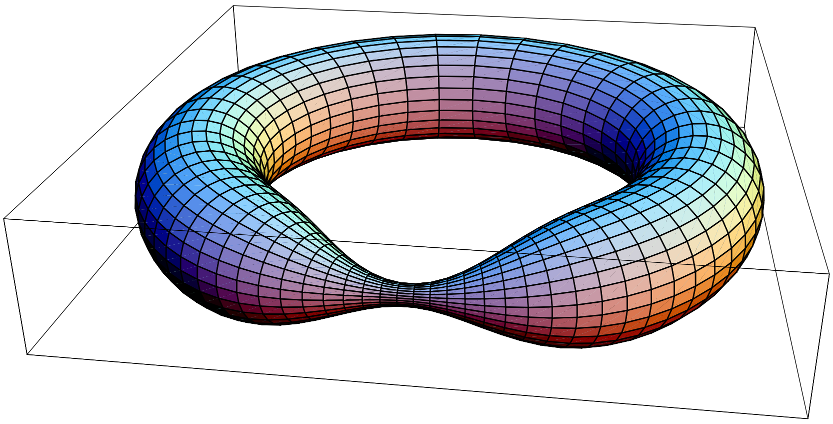


(a) Cone for the first parameterization of the parabolic surface. In this parameterization the line $B = \infty$ is left out.

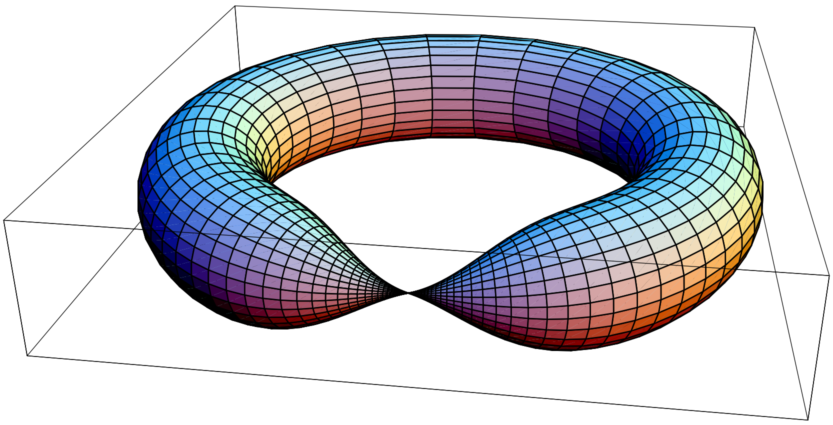


(b) Cone for the second parameterization of the parabolic surface. In this parameterization the line $D = \infty$ is left out.

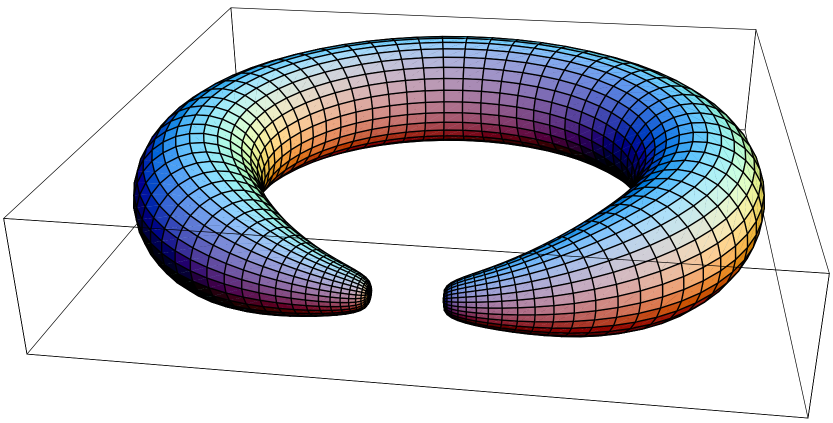
Figure 3.3: Every parabolic Monge-Ampère equation defines a singular surface in the fibers of the second order contact bundle. In suitable local coordinates this surfaces looks like a cone. In the closure of this cone in the fiber, the ends of the cone 'at infinity' are identified.



(a) A torus (hyperbolic Monge-Ampère equation)



(b) A constricted torus or pinched sphere (parabolic Monge-Ampère equation)



(c) A sphere (elliptic Monge-Ampère equation)

Figure 3.4: The deformation of a torus into a sphere. This deformation is exactly described by the equation $\det h = \epsilon$ where the parameter ϵ varies form -1 to 1. Note that the surfaces (torus, constricted torus, sphere) are embedded in the space of Lagrange planes $\Lambda(E, \sigma)$. This space is three dimensional, but not simply connected.

Chapter 4

Further research

In the previous chapter we have seen that for Monge-Ampère equations (a special class of second order partial differential equations in two independent variables) with constant coefficients we have a very nice description of the classes of contact equivalent equations. There are precisely three classes (elliptic, hyperbolic, parabolic), we have a fairly simple formula (see formula (3.6)) to determine the type of the equation and we have a constructive way of reducing a particular equation to a normal form (see the proof of theorem 3.1.4 in section 3.1).

In this chapter we will look at several possible generalizations of this theory. In particular we will consider increasing the number of variables to $n > 2$, allowing non-constant coefficients in the equations and allowing more general equations. It turns out that the theory and techniques from the previous chapters do not give any results for the more general problems directly, but the theory offers opportunities for more research.

4.1 More independent variables

In this section we will consider second order partial differential equations in n variables. Just as in chapter 3, we will only consider these equations in one fiber of the second order jet bundle or second order contact bundle.

For two independent variables we have defined the Monge-Ampère equations in (3.1). It turned out that all Monge-Ampère equations with contact coefficients are contact equivalent to a quasi-linear one and that there are exactly three orbits of the contact group. For one independent variable the situation is even more simple. The most general quasi-linear equation in one independent variable can be written as $\tilde{j}\tilde{h} + \tilde{l} = 0$ for a $\tilde{j} \neq 0$. Under a contact transformation this equation becomes $\tilde{j}(c + dh)(a + bh)^{-1} + \tilde{l} = 0$, or equivalently

$$(\tilde{j}d + b)h + c + a\tilde{l} = 0$$

on the open subset where $a + bh \neq 0$. We have to be a bit careful when $(a + bh) = 0$, but this can only happen for at most one value of h because not both a and b are zero. In the following discussion we will neglect these kind of conditions. The transformed equation is again a quasi-linear equation with coefficients $j = \tilde{j}d + b$ and $l = c + a\tilde{l}$. One can compare this with the new coefficients for a transformed equation in 2 variables (see formula (3.2)). We can easily see that we can always transform a quasi-linear equation to the standard form $h = 0$. Therefore the group of contact transformations acting on the quasi-linear (Monge-Ampère like) equations with constant coefficients in one independent variable has only one orbit.

An interesting question is what type of equations in n independent variables are contact equivalent to quasi-linear equations with constant coefficients for $n > 2$. The following lemma gives some control on the type of equations that are contact equivalent to a quasi-linear equation.

Lemma 4.1.1. *Every equation contact equivalent to a quasi-linear equation in a fiber is an equation of the form $j \det h + f(h)$, where $f(h)$ is a polynomial of degree $n - 1$.*

Proof. Every quasi-linear equation can be written as $\operatorname{tr}[\tilde{h}\tilde{k}] + \tilde{l}$ for a suitable $n \times n$-matrix $\tilde{k}$ and constant $\tilde{l}$. A general contact transformation transforms this equation into $\operatorname{tr}[(c+dh)(a+bh)^{-1}\tilde{k}]+\tilde{l}$ where the coefficients a, b, c, d are $n \times n$-matrices satisfying the conditions from theorem (2.5.1). The transformed equation can be written as

$$\operatorname{tr}[(c + dh)(a + bh)^{\mathrm{co}}\tilde{k}] + \tilde{l}\det(a + bh). \tag{4.1}$$

The determinant is a polynomial of degree n, the adjoint-matrix is a polynomial matrix of degree $n - 1$. For general n we have $\det(A + \lambda B) = \lambda^n \det B + \det A + g(A, B)$ where $g(A, B)$ is a polynomial in λ of order $n - 1$ and zero constant term. The transformed equation is therefore a polynomial of degree n in the coefficients of h and highest order term $(\operatorname{tr}[db^{\mathrm{co}}] + \det b)\det h$. $\square$

The equations given by polynomials of degree n in the coefficients of $\tilde{h}$ with highest order coefficient of type $\tilde{j}\det\tilde{h}$ we will call equations of *polynomial type*. It is not true that every equation of polynomial type is contact equivalent to a quasi-linear equation. We will prove this by considering for a given number of independent variables the dimension of the group of contact transformations, the dimension of the space of quasi-linear equations and the dimension of the space of all equations of polynomial type. In fact for a high number of independent variables the space of polynomial type equations is much larger that the space of equations contact equivalent to a quasi-linear equation.

Because the second order derivative matrix h is symmetric (the partial derivatives commute) this matrix has only $s = n(n + 1)/2$ independent coefficients. The space of quasi-linear equations with constant coefficients is a vector space over $\mathbb{R}$, its dimension we denote as $Q(n)$. The dimension of the space of polynomials of degree n with highest order term equal to a constant times $\det\tilde{h}$ we denote by $T(n)$. We write $D(n)$ for the dimension of the group of contact transformations at a point.

Lemma 4.1.2. *The group of transformations on Q_p induced by second order contact transformations has dimension $D(n) = 2n^2 + n + 1$. The dimension of the space of quasi-linear equations is equal to $Q(n) = s + 1$.*

For the polynomial type equations we have $T(n) = 1 + \sum_{k=0}^{n-1} F(s, k)$. Here $F(s, k)$ is the dimension of the vector space of homogeneous polynomials of degree k in s variables. For F we have the recursion relation $F(s, k) = F(s - 1, k) + F(s, k - 1)$ and the boundary conditions $F(s, 0) = 1$, $F(s, 1) = s$, $F(1, k) = 1$. For $T(n)$ we have the upper and lower bounds $T(n) \leq 1 + ns^{n-1}$ and $T(n) \geq \left(\frac{n-1}{2}\right)^{n-1}$.

Proof. The 4 matrix coefficients a, b, c, d have $4n^2$ degrees of freedom. The symmetric conditions (2.8) and (2.9) take each $n(n - 1)/2$ degrees, and the condition (2.10) takes $n^2 - 1$ degrees. The remaining degrees of freedom are therefore $4n^2 - (n^1 - 1 + n(n - 1)) = 2n^2 + n + 1$.

A quasi-linear equation has one degree of freedom for the constant term and s degrees of freedom for the terms linear in the second order derivatives. This implies $Q(n) = 1 + s$.

To see why the recursion relation holds, consider the following argument. Suppose the s variables are labelled $x_1, \ldots, x_s$. Then $F(s, k)$ is the number of monomials $x_{\pi(1)} \cdots x_{\pi(k)}$ where π ranges over all maps $\{1, \ldots, k\} \to \{1, \ldots, s\}$ for which $\pi(i) \leq \pi(i+1)$, $1 \leq i \leq k-1$. If $\pi(1) = 1$, then there are no restrictions on the values of $\pi(2)$ to $\pi(k)$, this gives $F(s, k - 1)$ possibilities. If $\pi(1) \neq 1$, so the first element of the monomial is not x_1, then the monomial contains only the $s - 1$ variables $x_2, \ldots, x_n$. This gives $F(s - 1, k)$ possibilities. The total number is therefore $F(s, k) = F(s, k - 1) + F(s - 1, k)$. The boundary conditions for $s = 1$ and $k = 1$ are obvious. There is only one constant monomial and therefore $F(s, 0) = 1$.

The formula for $T(n)$ in terms of the $F(s, k)$ is obvious from the definitions. To prove the upper bound on $T(n)$ note that the number $F(s, k)$ of monomials in s variables of length k is smaller that s^k. In the summation $\sum_{j=0}^{n-1} F(s, j)$ the term $F(s, n - 1)$ is the largest and therefore it follows that

$$T(n) \leq 1 + nF(s, n - 1) \leq 1 + ns^{n-1}.$$

For the lower bound note that the number of monomials in s variables of length k is larger then the number of monomials in s variables of length k where all the k variables are different. This means that for $k = n-1$

$$\begin{aligned} F(s,k) &\geq \binom{s}{k} = \frac{s(s-1)\cdots(s-k)}{k!} \\ &\geq \left(\frac{s-k}{k}\right)^k \geq \left(\frac{n-1}{2}\right)^{n-1}. \end{aligned}$$

The lower bound follows because $T(n) \geq F(s, n-1)$. $\square$

Remark 4.1.3 The dimension $D(n)$ of the group of contact transformations at a point is one higher than the dimension of the symplectic transformations. This is because the contact transformations only have to preserve the symplectic structure up to a nonzero factor.

In a similar way the dimension $Q(n)$ of the quasi-linear equations in n variables is one higher then the 'effective dimension'. This is because all equations can be multiplied by a nonzero scalar. $\oslash$

Using the previous lemma we can calculate the dimension of the relevant spaces for small n. The results are in the following table.

n	s	Q(n)	T(n)	D(n)
1	1	2	2	4
2	3	4	5	7
3	6	7	29	22
4	10	11	287	37
5	15	16	3877	56
6	21	22	65781	79

For $n = 1, 2$ the contact group has enough degrees of freedom to reduce all polynomial type equations to quasi-linear equations (= polynomials of degree 1). For $n = 2$ the contact group has 7 degrees of freedom, while in principle only one degree is needed to reduce the Monge-Ampère equations to quasi-linear ones. The remaining degrees of freedom can be used to further reduce the quasi-linear equations to some standard form. Although the dimension of the contact group is larger then the dimension of the Monge-Ampère equations, the action is not transitive because there are three orbits corresponding to the invariant type of the equation. For $n = 3$ the situation is more complicated. Although the dimension of the contact group (22) is in principle just enough to reduce all polynomial type equations to quasi-linear equations (29-7=22) this reduction is not possible. This can be proved by noting that the contact transformation already act on the quasi-linear equations. For $n = 4$ and higher we can see that the contact group has too little degrees of freedom and therefore not all polynomial equations with highest order term a multiple of $\det \tilde{h}$ are contact equivalent to quasi-linear ones.

We conclude these remarks by noting that reducing the transformed equation (4.1) to a polynomial will be difficult to do explicitly. For $n = 2$ we had fairly simple formulas for the determinant of the sum of matrices, the trace of an adjoint-matrix etc. (see appendix C.2). For $n > 2$ these formulas are not valid and it seems difficult to me to find similar formula's (even for $n = 3$) that can be used effectively. This makes the general reduction of the polynomial (4.1) into its homogenous parts a difficult problem.

4.2 Monge-Ampère equations with non-constant coefficients

So far we have only considered contact transformations in one point of the contact bundle. This corresponds to making an equation quasi-linear only in one point or working with equations with constant coefficients. In general the equations we want to solve have variable coefficients, i.e. the coefficients depend on the point in the base space and the first order derivatives. When making the Monge-Ampère equation quasi-linear in section 3.1, we considered these equations only in one point. The equation $j = 0$ indicating that the transformed equation was quasi-linear, was therefore an algebraic equation. If we want to make a Monge-Ampère equation quasi-linear in the neighborhood of a point, then the equation $j = 0$ is a partial differential equation for the contact transformation. This equation is in general a system non-linear second order partial differential equation and therefore a priori not simpler than the original equation.

In this section we will present a result of S. Lie that makes it easier to find a contact transformation making a Monge-Ampère equation quasi-linear. At the same time the result makes clear that solving the equation by reducing it to a quasi-linear equation does not make the problem less difficult. Lie states that knowing a 3-parameter family of solutions of the Monge-Ampère equation allows one make the equation quasi-linear. Lie mentioned this result in [20] and slightly reformulates the result in [21]. In these articles only the result is mentioned very briefly. These is no proof and no construction of the contact transformation mentioned. In the following paragraphs we give a proof of the statement of Lie and a construction of the contact transformation from the family of solutions that makes the equation quasi-linear. The idea for the proof and the construction is taken from [11, section 3.11].

We start with a general Monge-Ampère equation $j \det h + \operatorname{tr}[hk] + l = 0$, where we allow the j, k and l to depend on x, y and ξ. The equation is a partial differential equation in the two independent variables $x = (x_1, x_2)$ and the dependent variable y. We define B to be the base space of all points (x, y) and we define M to be the 1-jet bundle of B. Note that the projection $M \to B : (x, y, \xi) \to (x, y)$ is equal to the standard Legendre bundle. Suppose Φ is a contact transformation making the equation quasi-linear. The contact transformation Φ is a map between contact bundles that can be written locally as $M \to \tilde{M} : (x, y, \xi) \mapsto (\tilde{x}, \tilde{y}, \tilde{\xi})$. The projection $\tilde{M} \to \tilde{B} : (\tilde{x}, \tilde{y}, \tilde{\xi}) \mapsto (\tilde{x}, \tilde{y})$ onto the base space $\tilde{B}$ of $\tilde{M}$ is again a Legendre bundle. The fibers of the projection $(x, y, \xi) \to (x, y)$ are Legendre submanifolds and are mapped under the contact transformation to Legendre submanifolds. The tangent space $\{ (0, 0, \Xi) \in T_pP \mid \Xi \in \mathbb{R}^2 \}$ of such a Legendre manifold at the point $p = (x, y, \xi)$ is mapped to the tangent space $\{ (b\Xi, \xi^T b\Xi, d\Xi) \in T_{\tilde{p}}\tilde{M} \mid \Xi \in \mathbb{R}^2 \}$ at the point $\tilde{p} = \Phi(p)$. This tangent space is an n-dimensional Lagrange space and if b is invertible it can be written as

$$\{ (\tilde{X}, \tilde{\xi}^T \tilde{X}, db^{-1}X) \in T_{\tilde{p}}\tilde{M} \mid \tilde{X} \in \mathbb{R}^2 \} = \{ (\tilde{X}, \tilde{\xi}^T \tilde{X}, \tilde{h}\tilde{X}) \in T_{\tilde{p}}\tilde{M} \mid \tilde{X} \in \mathbb{R}^2 \}$$

with $\tilde{h} = db^{-1}$. This Lagrange space corresponds to the point $(\tilde{p}, \tilde{h}) = (\tilde{p}, db^{-1})$ in the fiber $\tilde{Q}_{\tilde{p}}$ of the second order contact bundle of $\tilde{B}$. Because b is invertible the projection from the transformed Legendre manifold to the base space $(\tilde{x}, \tilde{y})$ is a local diffeomorphism at the point $\tilde{p}$ and by theorem 2.2.1 we can write the Legendre submanifold as the 1-jet of a smooth function $\tilde{u}(\tilde{x})$. The second order derivative matrix $\tilde{h}$ of $\tilde{u}$ satisfies $\tilde{h} = db^{-1}$. The condition that the contact transformation makes the equation quasi-linear is $j = \tilde{j} \det d + \operatorname{tr}[db^{\mathrm{co}}\tilde{k}] + \tilde{l} \det b = 0$. If we multiply this by $\det b^{-1}$ this equation becomes $\tilde{j} \det db^{-1} + tr[db^{-1}\tilde{k}] + \tilde{l} = 0$, or

$$\tilde{j} \det \tilde{h} + \operatorname{tr}[\tilde{h}\tilde{k}] + \tilde{l} = 0.$$

So the function $\tilde{u}$ describing the transformed fiber $\{ (x, y, \xi) \mid (x, y) = \text{constant} \}$ must satisfy the original Monge-Ampère equation. We will now use this implication to find the desired contact transformation.

Suppose we have a 3-parameter family $\tilde{u}_\lambda$, $\lambda = (\lambda_0, \lambda_1, \lambda_2)$ of solutions of the Monge-Ampère equation with the following independence condition

$$\left.\frac{\partial}{\partial \lambda}\right|_{\lambda=0} \left(\tilde{u}_\lambda(\tilde{x}), \frac{\partial \tilde{u}_\lambda(\tilde{x})}{\partial \tilde{x}_1}, \frac{\partial \tilde{u}_\lambda(\tilde{x})}{\partial \tilde{x}_2} \right) \in \mathrm{GL}(3, \mathbb{R}). \tag{4.2}$$

We also assume that the family of solutions depends smoothly on λ in the sense that $(\lambda, \tilde{x}) \to \tilde{u}_\lambda(\tilde{x}) : \mathbb{R}^3 \times \mathbb{R}^n \to \mathbb{R}$ is a smooth map. Because $\tilde{M}$ is five dimensional and we have three parameters for our solution curves, the 1-jets of the solutions $\tilde{u}_\lambda(\tilde{x})$ locally fill up the entire contact manifold $\tilde{M}$. Moreover, the $j^1\tilde{u}_\lambda$ form a local Legendre fibration of $\tilde{M}$ with base space a neighborhood Λ of $\lambda = 0$ in the parameter space. To prove this consider the map

$$f : \Lambda \times \tilde{M} \to \mathbb{R} \times \mathbb{R}^n : (\lambda, (\tilde{x}, \tilde{y}, \tilde{\xi})) \mapsto (\tilde{u}_\lambda(\tilde{x}) - \tilde{y}, \frac{\partial \tilde{u}_\lambda(\tilde{x})}{\partial \tilde{x}} - \tilde{\xi}).$$

Because $\frac{\partial f}{\partial \lambda}$ is invertible in a neighborhood of $\lambda = 0$ (this follows from the independency condition on the family of solutions) we can use the implicit function theorem to conclude that there is a smooth function $\lambda = \psi(\tilde{x}, \tilde{y}, \tilde{\xi})$ such that $f(\psi(\tilde{x}, \tilde{y}, \tilde{\xi}), (\tilde{x}, \tilde{y}, \tilde{\xi})) = 0$ for all $(\tilde{x}, \tilde{y}, \tilde{\xi})$ in an open subset $\tilde{M}_0$ of $\tilde{M}$. For every parameter value $\lambda \in \Lambda$ the inverse image $\psi^{-1}(\lambda)$ is precisely the 1-jet of $\tilde{u}_\lambda(\tilde{x})$. The 1-jet $j^1\tilde{u}_\lambda$ is an integral manifold of the contact structure on $\tilde{M}$ and therefore a Legendre submanifold. We conclude that $\psi : \tilde{M}_0 \to \Lambda : (\tilde{x}, \tilde{y}, \tilde{\xi}) \mapsto \lambda$ is a Legendre fibration. This Legendre fibration is locally contact equivalent to the standard Legendre bundle $M \to B$ by theorem 2.6.6. This equivalence means that there is a contact transformation Φ from an open subset $M_0 \subset M$ to $\tilde{M}_0$ that maps the fibers $\{ (x, y, \xi) \mid (x, y) = \text{constant} \}$ to the solution curves $\tilde{u}_\lambda$. This contact transformation makes the equation quasi-linear.

We conclude that knowing a 3-parameter solution family makes the equation in principle quasi-linear. In the example below, we carry out the construction for a specific Monge-Ampère equation with variable constants. In the example it is easy to find the desired contact transformation. In general however it can require some work to find the desired contact transformation.

For analytic Monge-Ampere equations, i.e. the coefficients are analytic functions, there are existence theorems for 3-parameter families of solutions (see [22, Theorem 1.6]). Therefore an analytic Monge-Ampère equation can always be transformed to a quasi-linear equation.

Example 4.2.1 In this example we consider the Monge-Ampère equation with variable coefficients

$$f(x) \det h + g(x) \operatorname{tr}[hk] = 0 \tag{4.3}$$

for smooth functions f and g and a constant 2×2-matrix k. We will use the technique described in the previous paragraphs to find a contact transformation that makes this equation quasi-linear. A smooth 3-parameter solution to the differential equation (4.3) is given by

$$\tilde{u}_\lambda(\tilde{x}) = \lambda_0 + \lambda_1 \tilde{x}_1 + \lambda_2 \tilde{x}_2.$$

Take B to be $\mathbb{R}^2$ with local coordinates (x_1, x_2, y) and let M be the first order contact bundle of B. Let $\tilde{M}$ be defined as in the text above. The Legendre bundle corresponding to the fibering of $\tilde{M}$ by the solution curves $\tilde{u}_\lambda(\tilde{x})$ is given by the map $\tilde{M} \to \Lambda : (\tilde{x}, \tilde{y}, \tilde{\xi}) \mapsto (\lambda_0, \lambda_1, \lambda_2) = (\tilde{y} - \tilde{\xi}^T \tilde{x}, \tilde{\xi}_1, \tilde{\xi}_2)$. Because we want to make a contact transformation to the base space B we identify λ_0 with y and the pair (λ_1, λ_2) with x. We then have a Legendre bundle $\varpi : \tilde{M} \to B : (\tilde{x}, \tilde{y}, \tilde{\xi}) \to (x, y)$ with $x = \tilde{\xi}$ and $y = \tilde{y} - \tilde{\xi}^T \tilde{x}$. To make this into a bundle $M \to B : (x, y, \xi) \to (x, y)$ we use the construction from theorem 2.6.6. The tangent mapping $T_{\tilde{m}}\varpi : T_{\tilde{m}}\tilde{M} \to T_{\varpi(\tilde{m})}B$ is given by $(\tilde{X}, \tilde{Y}, \tilde{\Xi}) \mapsto (\tilde{\Xi}, \tilde{Y} - \tilde{\xi}^T \tilde{X} - \tilde{\Xi}^T \tilde{x})$, $\tilde{m} = (\tilde{x}, \tilde{y}, \tilde{\xi})$. The contact structure is given by $E_{\tilde{m}} = \{ (\tilde{X}, \tilde{Y}, \tilde{\Xi}) \in T_{\tilde{m}}\tilde{M} \mid \tilde{Y} = \tilde{\xi}^T \tilde{X} \}$ and so we find $(T_{\tilde{m}}\varpi)(E_{\tilde{m}}) = \{ (X, -\tilde{x}^T X) \in T_{\varpi(\tilde{m})}B \mid X \in \mathbb{R}^2 \}$. We see that the point $\tilde{m} = (\tilde{x}, \tilde{y}, \tilde{\xi}) \in \tilde{M}$ is mapped to the point $m = (x, y, \xi) = (\tilde{\xi}, \tilde{y} - \tilde{\xi}^T \tilde{x}, -\tilde{x})$ in M. The contact transformation found here is therefore $M \to \tilde{M} : (\tilde{x}, \tilde{y}, \tilde{\xi}) \mapsto (-\tilde{\xi}, \tilde{y} - \tilde{\xi}^T \tilde{x}, \tilde{x})$ and has coefficients $a = d = 0$ and $b = -c = -I$. The equation (4.3) is therefore transformed into the equation

$$-g(x) \operatorname{tr}[hk^{\text{co}}] + f(x) = 0.$$

⊘

4.3 Structure of the integral planes

Geometric problem Let (P, C) and (Q, D) be the first and second order contact bundles of a smooth 2-dimensional manifold Z. A partial differential equation on Q is a smooth hypersurface M

that is transversal to the fibers of the bundle $Q \to P$. We have $\dim Z = n = 2$, $\dim P = 2n+1 = 5$ and $\dim Q = n(n+1)/2 + 2n + 1 = 8$. For every point $m \in Q$ the second order contact structure defines a 5-dimensional linear subspace of the 8-dimensional tangent space T_mQ. The partial differential equation M is a 7-dimensional submanifold of Q and therefore has a 7-dimensional tangent space $TM \subset TQ$. Because M is transversal to the fibers $Q_p = (\pi_P^Q)^{-1}(p)$ and the second order contact structure is tangent to these fibers the subspaces D_m and T_mM intersect transversally at every point of M. This implies that $E_m = D_m \cap T_mM$ is a 4-dimensional linear subspace of T_mQ. The vector subbundle E defined by taking the intersection of D and TM is important to finding solutions of the partial differential equation because every solution of the partial differential equation is an integral manifold of E, that is at the same time contained in M. Conversely, every integral manifold of E contained in M can be written as the solution of the partial differential equation if the manifold projects nicely to the base manifold Z. The solutions of the partial differential equation are therefore related to the vector subbundle E.

For certain types of vector subbundles there are nice results on reducing the corresponding exterior differential system to a normal form (see [7, Chapter II]. In general however, it is not possible to reduce the vector subbundle to a standard form which we can solve easily. We therefore want to give a description of the integral planes, i.e. the integral elements of dimension two of the vector subbundle E. This description is a first step in finding and describing solutions of the partial differential equation. In [11, Chapter 4] a description is given of the structure of integral planes for the case of a single partial differential equation in two independent variables. We will present the main results of this description and give some comments on the approach of the proofs. We will also give an independent proof of some of these results using the theory on Monge-Ampère equations we developed in the previous chapter.

Formulation in differential forms We want to describe $\mathcal{I}_n(E)$, where $E = D \cap TM \subset TQ$. We will reformulate the problem in terms of exterior differential forms, this will make it easier to do computations. To be able to calculate the structure of the integral planes we introduce local coordinates for Q by taking the coordinates (x, y, z, p, q, r, s, t) as in example 1.1.5. Note that these are only local coordinates for Q, but because the description of the integral planes at a point $m = (x, y, z, p, q, r, s, t) \in M$ is a local problem we will not need any other coordinates. The contact structure D is given in these local coordinates by the common kernel of the three contact forms

$$\omega_0 = dz - pdx - qdy, \tag{4.4}$$

$$\omega_1 = dp - rdx - sdy, \quad \omega_2 = dq - sdx - tdy. \tag{4.5}$$

The forms ω_1 and ω_2 describe the second order structure, while ω_0 describes the first order structure. A tangent vector $(X, Y, Z, P, Q, R, S, T) \in T_mQ$ is therefore an element of the contact structure D if and only if

$$Z = pX + qY, \tag{4.6}$$

and

$$P = rX + sY, \quad Q = sX + tY. \tag{4.7}$$

Using these equations we can identify D_m with $\mathbb{R}^5$ using the map

$$(X, Y, R, S, T) \in \mathbb{R}^5 \mapsto (X, Y, pX + qY, rX + sY, sX + tY, R, S, T). \tag{4.8}$$

Having this description of the second order contact structure in coordinates for Q we turn to the partial differential equation defined by M. In local coordinates the submanifold M can be written in a neighborhood of m as the zero set of a smooth function f on Q. We can write this as

$$f(x, y, z, p, q, r, s, t) = 0. \tag{4.9}$$

To calculate E_m we need to describe the tangent space of M. It is clear that $H = (X, Y, R, S, T) \in D_m \subset T_m Q$ is in the tangent space of M if and only if $H \in \ker df(m)$. This is equivalent to $\frac{d}{dt}|_{t=0} f(m + tH) = 0$, or

$$f_x X + f_y Y + f_z Z + f_p P + f_q Q + f_r R + f_s S + f_t T = 0. \tag{4.10}$$

Here we have written f_j for the derivative of f with respect to the coordinate j at the point m. We eliminate Z, P and Q using the equations (4.7) and find that the equation above is equivalent to

$$\tilde{f}_x X + \tilde{f}_y Y + f_r R + f_s S + f_t T = 0 \tag{4.11}$$

with

$$\tilde{f}_x = f_x + p f_z + r f_p + s f_q, \quad \tilde{f}_y = f_y + q f_z + s f_p + t f_q. \tag{4.12}$$

Because M is transversal to the fibers of $Q \to P$ we know that the dependence of f on the fiber coordinates r, s, t is non-trivial, i.e. $(f_r, f_s, f_t) \neq 0$. By means of a suitable rotation of the (x, y) coordinates (this rotation is a contact transformation) we can arrange that $f_t \neq 0$ and by scaling f we can even arrange that $f_t = 1$. This makes it possible to identify E_m with the four-dimensional linear space $\mathbb{R}^4$ with coordinates (X, Y, R, S) using the map

$$\begin{aligned} (X, Y, R, S) \in \mathbb{R}^4 \mapsto (X, Y, pX + qY, rX + sY, sX + tY, R, S, T) \in E_m, \\ T = f_t^{-1}(-\tilde{f}_x X - \tilde{f}_y Y - f_r R - f_s S). \end{aligned} \tag{4.13}$$

The integral planes we are looking for are the 2-dimensional linear subspaces of E_m for which the Lie brackets modulo the contact structure vanish. This geometric condition can be formulated in terms of the differential forms by saying that a 2-dimensional linear subspace I of E_m is an integral element of E if $d\omega_j|_{I \times I} = 0$, $0 \leq j \leq 2$. The exterior derivatives of the differential forms are

$$d\omega_0 = dx \wedge dp + dy \wedge dy, \tag{4.14}$$

$$d\omega_1 = dx \wedge dr + dy \wedge ds, \quad d\omega_2 = dx \wedge ds + dy \wedge dt. \tag{4.15}$$

The description of the integral planes is therefore reduced to the study of the three anti-symmetric forms $d\omega_j$, $0 \leq j \leq 2$ above in combination with the identification (4.13).

The first order conditions The differential form ω_0 corresponds to the first order contact structure. The conditions the exterior differential of first order contact structure imposes on the integral planes are automatically satisfied for the second order integral elements. We formulate this more precisely in the following lemma.

Lemma 4.3.1. *For the first order contact form we have $d\omega_0|_{D_m \times D_m} = 0$. In particular the two form $d\omega_0$ vanishes already on every 2-dimensional linear subspace of E_m.*

Proof. By definition the contact forms ω_1 and ω_2 vanish on D_m. This allows us to solve for dp and dq in the expressions for ω_1 and ω_2. If we substitute the expressions for dp and dq in $d\omega_0$ we find

$$\begin{aligned} d\omega_0|_{D_m \times D_m} &= dx \wedge (rdx + sdy) + dy \wedge (sdx + tdy) \\ &= dx \wedge (sdy) + dy \wedge (sdx) = 0. \end{aligned}$$

Because $E_m \subset D_m$ the second statement is obvious. □

The second order conditions The second order contact structure is given by the two differential forms ω_1 and ω_2. The condition that a 2-dimensional linear subspace I of E_m is an integral element of E_m is equivalent to the condition that I is an integral element for both ω_1 and ω_2. This integral condition means that the linear space I should be isotropic with respect to the bilinear forms $\sigma = d\omega_1$ and $\tau = d\omega_2$, i.e. we must have $I \subset I^\sigma$ and $I \subset I^\tau$. The structure of the integral planes is therefore completely determined by the two anti-symmetric bilinear forms σ and τ.

Direct calculation The bilinear form σ is non-degenerate on E_m. If we identify the space E_m with the space (X, Y, R, S) and identify this with $\mathbb{R}^4$, then $\sigma = dx \wedge dr + dy \wedge ds$ is just the standard symplectic form. Therefore a 2-dimensional linear space I is isotropic with respect to σ if and only if it is a Lagrange plane of the symplectic structure defined by σ. The integral elements we are looking for are therefore a subset of $\Lambda(E_m, \sigma)$. The condition that an element I of (E_m, σ) is also τ isotropic is an extra condition.

Theorem 4.3.2. *Let f, m and E be defined as above. The structure of the integral planes at the point m is determined by the type of the partial differential equation $f = 0$ at m. In particular*

i) $\mathcal{I}_2(E)_m$ is a sphere in $G_2(E)_m$, if f is elliptic at m,

ii) $\mathcal{I}_2(E)_m$ is a torus in $G_2(E)_m$, if f is hyperbolic at m,

iii) $\mathcal{I}_2(E)_m$ is a constricted torus in $G_2(E)_m$, if f is parabolic at m .

Proof. We will prove the theorem by showing that the condition that I is both σ and τ isotropic defines an equation on the space $\Lambda(E_m, \sigma)$. This equation is given in one of the standard charts by a quasi-linear equation. We know from the previous section that all these quasi-linear equations are contact equivalent and that the three equivalence classes elliptic, hyperbolic and parabolic define a sphere, torus and constricted torus, respectively. The equation defined by the isotropy condition of τ is essentially the linearization of the function f and therefore related to the type of f.

We consider the open chart of the space $\Lambda(E_m, \sigma)$ defined by the planes for which $dX \wedge dY \neq 0$. On this affine part we can write every Lagrange plane as

$$\{ (X, Y, R, S) \mid (R, S)^T = h(X, Y)^T \} \tag{4.16}$$

for a symmetric 2×2-matrix h. The form τ defines an equation for the matrix h that we will now work out. We can rewrite the anti-symmetric bilinear form τ restricted to E_m, with the identifications we have made so far, as

$$\begin{aligned}
\tau|_{E_m} &= dx \wedge ds + dy \wedge dt \\
&= dx \wedge ds + dy \wedge f_t^{-1}(-\tilde{f}_x dx - \tilde{f}_y dy - f_r dr - f_s ds) \\
&= dx \wedge ds + f_t^{-1}(\tilde{f}_x dx \wedge dy - f_r dy \wedge dr - f_s dy \wedge ds).
\end{aligned}$$

The condition that τ restricted to the plane defined by the matrix h is zero is equivalent to the vanishing of τ on any pair of linearly independent vectors in the plane. We choose for these vectors $X_1 = (\alpha, 0, h_{11}\alpha, h_{21}\alpha)$ and $X_2 = (0, \beta, h_{12}\beta, h_{22}\beta)$. Then

$$\tau(X_1, X_2) = \alpha h_{22}\beta + 0 - 0 - \beta f_t^{-1}(-\tilde{f}_x\alpha - f_r h_{11}\alpha - f_s h_{21}\alpha)$$

and

$$f_t\tau(X_1, X_2) = \alpha\beta(h_{22} f_t + \tilde{f}_x + f_r h_{11} + f_s h_{21}).$$

This implies that the plane defined by h is τ isotropic if and only if the equation

$$\mathrm{tr}[h \begin{pmatrix} f_r & f_s/2 \\ f_s/2 & f_t \end{pmatrix}] + \tilde{f}_x = 0 \tag{4.17}$$

is satisfied. The equation is a quasi-linear equation (in particular a Monge-Ampère equation). The type of this equation is also equal to the type of the linearization of the partial differential equation $f = 0$ (see example 3.2.1). The surface described by the equation is a torus, sphere or constricted torus (see section 3.3) depending on the sign of $\det h = f_r f_t - f_s^2$. In this way we have proved the geometry of the surfaces of integral planes in a way independent of the characterization using the map A to be described in the next paragraphs. □

Complex structure of $\mathbf{E_m}$ The classification of structures of integral planes corresponds to the classification of a pair (σ, τ) of anti-symmetric bilinear forms on a four-dimensional linear space E of which one of the forms is non-degenerate, i.e. defines a symplectic structure. If we assume σ is non-degenerate, then σ defines a bijective linear map $E \to E^*$ by taking $e \in E \mapsto \sigma(e, \cdot)$. We can then define $A = \sigma^{-1} \circ \tau$ as a linear mapping from E to E. This map is σ-symmetric in the sense that $\sigma(A(x), y) = \sigma(x, A(y))$ for all elements $x, y \in E$. For a linear map that is symmetric with respect to a symplectic structure we have the following general lemma.

Lemma 4.3.3. *Let E be a finite-dimensional vector space and σ a symplectic form on E. If A is a σ-symmetric linear map from E to E then*

i) the (complex) eigenvalues of A have even algebraic multiplicity,

ii) the eigenspaces of A 'preserve' the symplectic structure of σ. For each eigenspace E_λ the restriction of σ to $E_\lambda \times E_\lambda$ is a symplectic form.

There is a unique monic polynomial $q(\lambda)$ of degree n, called the Pfaffian!of a linear map of A, such that $\det(A - \lambda I) = q(\lambda)^2$.

On each of the eigenspaces E_λ the map A can be written as $\lambda I + N_\lambda$, with N_λ is a σ-symmetric nilpotent matrix. It turns out that since $\dim E_\lambda$ is either 2 or 4 there are only 3 possibilities for the nilpotent map N_λ. If $\dim E_\lambda = 2$ then N_λ must be trivial, if $\dim E_\lambda = 4$ then N_λ is either trivial or we can choose a suitable basis on which σ is equal to the standard symplectic form and N_λ has the matrix representation

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.18}$$

The combination of these results leads to the following result for a four dimensional symplectic vector space (E, σ) with a σ-symmetric map A.

Lemma 4.3.4. *Let (E, σ) be a symplectic vector space of dimension 4 and A a σ-symmetric linear map $E \to E$. Then there are four (mutually exclusive) cases for the eigenvalues of A*

i) The matrix A has two distinct real eigenvalues, each with multiplicity 2.

ii) The matrix A has two complex eigenvalues λ and $\bar{\lambda}$, each with multiplicity two.

iii) The matrix A has one real eigenvalue with multiplicity four and a non-trivial nilpotent part.

iv) The matrix A has one real eigenvalue with multiplicity four and a trivial nilpotent part, i.e. on a suitable basis A is a multiple of the identity.

The last case cannot occur in the case of a second order partial differential equation because this would imply $\tau = \lambda\sigma$. This cannot occur because we have arranged that $f_t \neq 0$. Therefore there are only three possibilities for the matrix $A = \sigma^{-1} \circ \tau$. This leads to the following characterization of the pair (σ, τ) on E_m.

Theorem 4.3.5 (Lemma 4.8 and Theorem 4.9 in [11]). *Let E, m, σ and τ be defined as above. Then we have the following three possible situations*

i) The matrix A has two real distinct eigenvalues λ, μ, each with multiplicity 2. The eigenspaces E_λ and E_μ are two dimensional real linear subspaces of E. For every linear subspace l of E_λ and every linear subspace k of E_μ the two dimensional linear subspace $I = k \oplus l$ is an element of $\mathcal{I}_2(E)_m$. All integral planes are of the form $k \oplus l$, with l and k 1-dimensional linear subspaces of E_λ and E_μ.

ii) The complex eigenvalues of A are λ and $\bar{\lambda}$, each with multiplicity 2. The eigenspaces E_λ and $E_{\bar{\lambda}}$ are complex two-dimensional linear subspaces of $E \otimes \mathbb{C}$. For every complex 1-dimensional linear subspace of E_λ we have that $I = (l \oplus \bar{l}) \cap E$ is a real 2-dimensional subspace of E_m that is both σ and τ isotropic. Again, all elements of $\mathcal{I}_2(E)_m$ are of this form.

iii) The matrix A has one real eigenvalue λ. On a suitable basis we have that $\rho = \tau - \lambda\sigma$ is a degenerate two-form that has the form (4.18). We define F as $F = \ker \rho$, i.e. F is the linear subspace of E spanned by the first two basis vectors of a basis in which ρ takes the form (4.18). For every 1-dimensional subspace l of F and every 1-dimensional linear subspace k of l^σ such that $k \neq l$, the 2-dimensional plane $I = k \oplus l$ is an element of $\mathcal{I}_2(E)_m$.

The different cases in the theorem above are determined by the eigenvalues of A. Since $\det(A - \lambda I) = q(\lambda)^2$ the type is determined by the Pfaffian $q(\lambda)$. Under the assumptions of theorem 4.3.5 we have $q(\lambda) = \lambda^2 + f_s\lambda + f_r f_t$. The determinant of this quadratic form in λ is $\Delta = f_s^2 - 4 f_r f_t$. In this way we see again that the type of f at m is directly related to the structure of the integral planes in E_m. Not surprisingly, case i) corresponds to a hyperbolic partial differential equation, case ii) to an elliptic one and finally case iii) to a parabolic equation. Using the structure of the integral planes described in the above theorem one can show that the surfaces of integral planes are diffeomorphic to a torus, sphere and constricted torus in case i), ii) and iii), respectively.

Final notes In the case of two independent variables and one dependent variable we have given a complete and quite transparent description of the integral planes at a point of a general partial differential equation. The completely different structure of the integral planes in the elliptic and hyperbolic case is a confirmation of the fact that solutions of elliptic and hyperbolic equations have very different characteristics.

However there is still a lot of work to do. First of all we should have some techniques or theorems on how we can use the structure of the integral planes at a point to conclude something about the solutions of the partial differential equation. These solutions are always defined on an open subset and never in only one point. Secondly, we want to expand our description of the integral planes to more general situations. We want to allow more independent variables and more than one dependent variable (systems of partial differential equations). Also we would like to generalize our theory as much as possible to higher order partial differential equations.

4.4 Relation of the symplectic group to the surfaces of integral planes

The contact transformations induce symplectic transformations on the fibers of the second order contact bundle. We will give another description of the structure of the integral planes in the case of an hyperbolic equation, in terms of the symplectic transformations. For a hyperbolic equation we can always choose a suitable coordinate system, such that the surface of integral planes is given by the closure of the Lagrange planes given by

$$\{ \begin{pmatrix} a\alpha \\ b\beta \\ \alpha \\ \beta \end{pmatrix} \mid \alpha, \beta \in \mathbb{R} \}$$

for $a, b \in \mathbb{R}$. We can write the closure of these Lagrange planes as

$$\{ \begin{pmatrix} (\sin\theta)\alpha \\ (\sin\phi)\beta \\ (\cos\theta)\alpha \\ (\cos\phi)\beta \end{pmatrix} \mid \alpha, \beta \in \mathbb{R} \}$$

for $\theta, \phi \in [0, \pi]$. This shows again that the surface of integral planes is diffeomorphic to a torus. We can define the following elements of the symplectic group

$$t_{\theta,\phi} = \begin{pmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & \cos\phi & 0 & -\sin\phi \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & \sin\phi & 0 & \cos\phi \end{pmatrix} \tag{4.19}$$

for $\theta, \phi \in [0, 2\pi]$. We denote the subgroup of the symplectic transformations generated by these elements as H. The multiplication in H is given by $t_{\theta,\phi} t_{\theta',\phi'} = t_{\theta+\theta',\phi+\phi'}$.

We choose as a base point the Lagrange plane M_2, which is given by

$$M_2 = \{ \begin{pmatrix} 0 \\ 0 \\ \alpha \\ \beta \end{pmatrix} \mid \alpha, \beta \in \mathbb{R} \}.$$

The subgroup H acts transitively on the surface of integral planes. The subgroup also leaves the surface invariant. From the general theory of actions of Lie groups on smooth manifolds it follows that the surface of integral planes is diffeomorphic to the subgroup H modulo the stabilizer group of M_2. This stabilizer group is $\{ \pm I \}$ and therefore the surface of integral planes is a torus.

Perhaps the elliptic case can also be described in this way. To find such a description one only needs to find enough symplectic transformations that leave the surface invariant, such that the group generated by these elements acts transitively on a base point in the surface. Such transformations can be found by trial and error, but perhaps the Lie algebra of the group of contact transformations can be useful. The parabolic case can not be described in this way, because in the parabolic case the surface has one singular point. If the description of the elliptic and hyperbolic case is successful, then maybe we can use the same techniques to describe the surfaces of integral planes for more independent variables in the elliptic and hyperbolic cases.

Appendix A

Vector subbundles

Let M be a smooth manifold and let E be an n-dimensional vector subbundle of TM. This means that for each $m \in M$ the linear subspace E_m is an n-dimensional linear subspace of $T_m M$. If $\dim E_m$ is constant and $\dim E_m = s$, then we call s the dimension of the vector subbundle. A vector subbundle of TM is sometimes called a *distribution* or *multi-vector field* on M.

A.1 Pfaffian forms

It is sometimes useful to describe vector subbundles in a different way. A Pfaffian form is a smooth differential form of degree one. Suppose we have a system of n linear independent Pfaffian forms $\omega_1, \ldots, \omega_n$ on the manifold M. Linear independent means that for all $m \in M$ the forms $\omega_{1,m}, \ldots, \omega_{n,m}$ on the tangent space $T_m M$ are linear independent. We can now define for all $m \in M$ the subspace $E_m = \{ X \in T_m M \mid \omega_i(X) = 0,\, i = 1, \ldots, n \} = \cap_{i=1}^{n} \ker \omega_i$. These subspaces define a smooth vector subbundle E of TM. Because the differential forms are linear independent, the linear subspace E_m has dimension $\dim M - n$ for all points in M.

The converse is true only locally. If E is a codimension c vector subbundle of TM then locally we can write E as the kernel of a system of c linear independent Pfaffian forms.

Definition A.1.1. A pair (M, E) of a smooth manifold M and a smooth vector subbundle is called a *Pfaffian system*.

A.2 Integral manifolds

Definition A.2.1. A submanifold U of M is called an *integral manifold* of E if for all $m \in M$ we have $T_m U \subset E_m$. A vector subbundle E is called *integrable* if every $m \in M$ is contained in an integral manifold U of dimension $\dim E_m$.

Example A.2.2

i) The tangent space TM of a smooth manifold M is an integrable subbundle.

ii) Every 1-dimensional vector subbundle of TM is a vector field on M. From the theory of ordinary differential equations it follows that every 1-dimensional vector subbundle is integrable (the integral manifolds being the integral curves of the corresponding vector field).

iii) Not every distribution is integrable. Consider for example $M = \mathbb{R}^3$ with coordinates (x, y, z). On M we define the vector fields $X = \partial_x - y\partial_z$, $Y = \partial_y$ and $Z = \partial_z$. For $m \in M$ we take E_m to be the linear subspace of $T_m M$ spanned by the vectors X_m and Y_m. It is clear that E is a 2-dimensional vector subbundle of TM.

Now suppose that U is an integral manifold for E with $\dim U = 2$ and $0 \in U$. Because U is an integral manifold it is clear that both $X' = X|_U$ and $Y' = Y|_U$ are vector fields on U. We know that the Lie bracket of X and Y is a vector field that is also tangent to U. For m in U we have that $X_m \in T_mU$, $Y_m \in T_mU$ and $[X,Y]_m \in T_mU$. But $[X,Y]_m = (\partial_z)_m = Z_m$, so this means that $X_m, Y_m, Z_m \in T_mU$. Because T_mU is a linear subspace it follows that $T_mU = T_mM$ in contradiction with the fact that $\dim U = 2$.

Note that the reason for the non-integrability of E was the fact that the Lie bracket $[X,Y]$ was nonzero. If we take the distribution spanned by $X = \partial_x$ and $Y = \partial_y$ then the commutator of X and Y is zero. This distribution is integrable as one can easily check. The integral manifolds are just the planes on which z is constant.

$\oslash$

We want to know whether a distribution E is integrable or not. In the example above we have seen that the Lie bracket of the vectors in the distribution is important. It turns out that not the Lie bracket is the important map, but another closely related map called the Lie brackets modulo the subbundle.

Lemma A.2.3. *Let E be a distribution on the manifold M. Let $m \in M$ and X and Y be vector fields in E, so $[X,Y]_m$ is an element of T_mM. The equivalence class $[X,Y]_m + E_m$ in T_mM/E_m depends only on X_m and Y_m. This defines an antisymmetric bilinear mapping $E_m \times E_m \to T_mM/E_m$, which we denote by $X, Y \to [X,Y]_{/E_m}$ and which we call the* Lie brackets modulo the subbundle.

Proof. The only thing we need to proof is that $[X,Y]_{/E_m}$ depends only on X_m and Y_m and not on the choice of vector fields X and Y. This is not at all obvious because in general the Lie bracket $[X,Y]$ does depend on the first derivatives of X and Y.

Suppose we choose vector fields X, Y, X', Y' in E such that $X_m = X'_m$ and $Y_m = Y'_m$. It is then sufficient to prove that $[X,Y]_m - [X',Y']_m \in E_m$. Locally we can choose a basis of vector fields $V_1, \ldots, V_n$ for E (so E_m is spanned by $(V_1)_m, \ldots, (V_n)_m$). We can then write

$$X' = X + \sum_j f^j V_j, \quad Y' = Y + \sum_j g^j V_j$$

for unique smooth functions $f^1, \ldots, f^n$ and $g^1, \ldots, g^n$ on a neighborhood of m. Because $X_m = X'_m$ and $Y_m = Y'_m$ it follows that $f^j(m) = 0$ and $g^j(m) = 0$ for $1 \leq j \leq n$. We can now calculate $[X,Y]_m - [X',Y']_m$ in terms of the functions f^j and g^j and the vector fields V_j. In the following formula we use the summation convention.

$$\begin{aligned}
[X,Y]_m - [X',Y']_m &= -[f^jV_j, Y]_m - [X, g^jV_j]_m - [f^jV_j, g^iV_i]_m \\
&= -f^j(m)[V_j, Y]_m + Y(f_j)(m)(V_j)_m - g^j(m)[X, V_j]_m - X(g^j)(m)(V_j)_m \\
&\quad - f^j(m)g^i(m)[V_j, V_i]_m + g^i(m)V_i(f^j)(m)(V_j)_m + f^j(m)V_j(g^i)(m)(V_i)_m \\
&= Y(f_j)(m)(V_j)_m - X(g_j)(m)(V_j)_m \\
&= (df^j)_m(Y)(V_j)_m - (dg^j)_m(X)(V_j)_m \in E_m.
\end{aligned}$$

This proves that $[X,Y]_{/E_m}$ depends only on X_m and Y_m and not on the specific choice of X and Y. $\square$

Note that the mapping $[\cdot,\cdot]_{/E_m}$ from $E_m \times E_m$ to T_mM/E_m depends on E and not only on the linear subspace E_m.

Definition A.2.4. An *integral element* of E at the point m is a linear subspace I of E_m such that the Lie brackets modulo the subbundle are identically zero on $I \times I$.

For a given Pfaffian system (M, E) we write $\mathcal{I}_k(E)_m$ for the space of k-dimensional integral elements at m. The union of all k-dimensional elements over all points $m \in M$ forms a bundle over M, which we denote by $\mathcal{I}_k(E)$. Even for smooth Pfaffian systems the structure of the bundles $\mathcal{I}_k(E) \to M$ can get complicated. We refer to [7, Chapter 3] for theory on the structure of these spaces of integral elements.

If the subbundle E is defined by the Pfaffian system $\omega_1, \dots, \omega_p$, then the condition that I is an integral element is equivalent to $d\omega_i|_{I \times I} = 0$ for $1 \leq i \leq p$. See [7, paragraphs 2.1 and 3.1]. If E is an integrable subbundle then locally E is equal to the tangent space of a manifold U and $E_m = T_m U$. Since the tangent space of a manifold is closed under Lie brackets, it follows that E_m is an integral element. So if E is integrable then every subspace E_m is an integral element. The converse is also true and this statement is known as the Frobenius theorem.

Theorem A.2.5 (Frobenius)**.** *Let M be a smooth manifold and E be a smooth vector subbundle of TM. Then E is integrable if and only if for every $m \in M$ the space E_m is an integral element of E.*

Proof. The proof of the first implication was already sketched in the paragraph before. For the reverse implication we refer to [10, Theorem 3.1.1]. □

Remark A.2.6 The formulation in terms of vector subbundles of the tangent space is quite natural when working in the first and second order contact bundle. An equivalent formulation, but one that allows for generalizations more easily, can be given in terms of exterior differential systems. The exterior differential systems are more general than Pfaffian systems, because we can allow differential forms of any order (not only 1-forms). The exterior differential systems were introduced by Cartan in [8]. An overview of differential systems and their relation to Pfaffian systems is given in [7]. ⊘

Appendix B

Symplectic geometry

B.1 Symplectic vector spaces

Definition B.1.1. Let E be a finite-dimensional vector space. A *symplectic form* on E is an antisymmetric bilinear non-degenerate form on E. Non-degenerate means that if $\sigma(x, y) = 0$ for all $y \in E$ then x must be zero. The pair (E, σ) is called a *symplectic vector space*.

For a linear subspace L of E we define the orthogonal complement L^σ to be the linear space of elements $x \in E$ for which $\sigma(x, y) = 0$ for all $y \in L$. A linear subspace L is called isotropic if $L \subset L^\sigma$. An n-dimensional isotropic subspace of E is called a Lagrange plane. We write $\Lambda(E, \sigma)$ for the set of all Lagrange planes in E.

Proposition B.1.2. *Let (E, σ) be a symplectic vector space. The set of Lagrange planes is an $n(n+1)/2$-dimensional manifold.*

Proof. We will prove this by covering $\Lambda(E, \sigma)$ with open sets diffeomorphic to $\mathrm{Symm}^2(\mathbb{R}^n)$, the space of symmetric bilinear forms in $\mathbb{R}^n$. We will not give all details in the proof.

Suppose $M \in \Lambda(E, \sigma)$. Choose a Lagrange plane $M^\perp$ transversal to M (so $M \cap M^\perp = \{0\}$). We write $\Lambda^0(E, \sigma, M^\perp)$ for the set of all Lagrange planes transversal to $M^\perp$. Note that $M \in \Lambda^0(E, \sigma, M^\perp)$ by definition. If $L \in \Lambda^0(E, \sigma, M^\perp)$ then we can write L as

$$\{ x + A_L(x) \in E \mid x \in M \} \tag{B.1}$$

for a unique linear map $A_L : M \to M^\perp$. This linear map A_L in turn defines a unique bilinear form β_L on M by $\beta_L(x, y) = \sigma(A_L(x), y)$.

One can check that the mapping $L \to \beta_L$ is a diffeomorphism from the n-dimensional planes transversal to M onto the bilinear forms on M. If L is a Lagrange plane, then $\sigma(x, y) = 0$ for all $x, y \in L$. This means in particular that $\sigma(x + A_L(x), y + A_L(y)) = 0$ for all $x, y \in M$. Because $x, y \in M$, $A_L(x), A_L(y) \in M^\perp$ and $M, M^\perp$ are Lagrange planes, it follows that $\sigma(x, y) = 0$ and $\sigma(A_L(x), A_L(y)) = 0$. The condition that L is a Lagrange plane therefore reduced to $\sigma(x, A_L(y)) + \sigma(y, A_L(x)) = \beta_L(x, y) - \beta_L(y, x) = 0$. So if L is a Lagrange plane then β_L is symmetric. It is not difficult to show that every symmetric bilinear form β defines a Lagrange plane L such that $\beta = \beta_L$.

This means that the map $L \to \beta_L$ maps $\Lambda^0(E, \sigma, M)$ onto $\mathrm{Symm}^2(M)$. Because M is an n-dimensional plane in the linear space E, the hyperplane M is diffeomorphic to $\mathbb{R}^n$. This proves that the open neighborhood $\Lambda^0(E, \sigma, M^\perp)$ of the point M is diffeomorphic to $\mathrm{Symm}^2(\mathbb{R}^n)$. $\square$

B.2 Standard symplectic space

We take $E = \mathbb{R}^{2n}$ with standard coordinates $x = (x_1, \ldots, x_{2n}) = (p, q) = (p_1, \ldots, p_n, q_1, \ldots, q_n)$ and standard inner product $\langle x \, , \, y \rangle = \sum_{i=1}^{2n} x_i y_i$. We define complex conjugation on E by the map

$I : E \to E : (p, q) \to (-q, p)$. The *standard symplectic form* on E is given in terms of the complex conjugation and the inner product by $\tau_n(x, y) = \langle Ix\,,\, y\rangle$ for all $x, y \in E$. The pair $(\mathbb{R}^{2n}, \tau_n)$ is called the standard symplectic vector space of dimension $2n$.

Definition B.2.1. Let (E, σ) and (E', σ') be symplectic vector spaces. A linear map $A : E \to E'$ is called a *symplectic map* if A preserves the symplectic structures. This means that $A^*\sigma' = \sigma$. Because σ is non-degenerate, the symplectic mapping A is always injective. A surjective symplectic mapping (that is automatically bijective) is called a *symplectic isomorphism*.

Theorem B.2.2. *Every symplectic vector space (E, σ) of dimension $2n$ is isomorphic with a symplectic isomorphism to the standard symplectic space $(\mathbb{R}^{2n}, \tau_n)$.*

Proof. See [10, Theorem 3.4.2]. □

Because every symplectic space (E, σ) is isomorphic to the standard symplectic space, we only need to study the symplectic properties of standard symplectic space. For the specific case $n = 2$ we will work out the coordinate charts for the manifold of Lagrange planes in detail.

We now consider the special case $n = 2$. The symplectic form is given by $\sigma(x, y) = x_1y_3 + x_2y_4 - x_3y_1 - x_4y_2$. We define the four standard Lagrange planes by

$$M_1 = \{ \begin{pmatrix}\alpha\\ \beta\\ 0\\ 0\end{pmatrix} \mid \alpha, \beta \in \mathbb{R} \}, \qquad M_2 = \{ \begin{pmatrix}0\\ 0\\ \alpha\\ \beta\end{pmatrix} \mid \alpha, \beta \in \mathbb{R} \},$$

$$M_3 = \{ \begin{pmatrix}\alpha\\ 0\\ 0\\ \beta\end{pmatrix} \mid \alpha, \beta \in \mathbb{R} \}, \qquad M_4 = \{ \begin{pmatrix}0\\ \alpha\\ \beta\\ 0\end{pmatrix} \mid \alpha, \beta \in \mathbb{R} \}.$$

The open sets $\Lambda^0(E, \sigma, M_j)$ cover the manifold $\Lambda(E, \sigma)$ (this is proved in [3]). We have already seen that each open subset $\Lambda^0(E, \sigma, M_j)$ is diffeomorphic with $V = \mathrm{Symm}^2(\mathbb{R}^2)$. This diffeomorphism still depends on the choice of a diffeomorphism $M_j \to \mathbb{R}^n$. Here we will make a specific choice for each of the four standard Lagrange planes. We take

$$M_1 \to \mathbb{R}^2 : \begin{pmatrix}\alpha\\ \beta\\ 0\\ 0\end{pmatrix} \mapsto \begin{pmatrix}\alpha\\ \beta\end{pmatrix}, \qquad M_2 \to \mathbb{R}^2 : \begin{pmatrix}0\\ 0\\ \alpha\\ \beta\end{pmatrix} \mapsto \begin{pmatrix}\alpha\\ \beta\end{pmatrix},$$

$$M_3 \to \mathbb{R}^2 : \begin{pmatrix}\alpha\\ 0\\ 0\\ \beta\end{pmatrix} \mapsto \begin{pmatrix}\alpha\\ \beta\end{pmatrix}, \qquad M_4 \to \mathbb{R}^2 : \begin{pmatrix}0\\ \alpha\\ \beta\\ 0\end{pmatrix} \mapsto \begin{pmatrix}\alpha\\ \beta\end{pmatrix}.$$

These choices determine diffeomorphisms $L_j : \mathrm{Symm}^2(\mathbb{R}^2) \to \Lambda^0(E, \sigma, M_j)$, $1 \le j \le 4$. We will construct L_1 explicitly and give for L_2, L_3 and L_4 only the final result.

The set of 2-dimensional planes transversal to M_1 is given by all planes L of the form

$$L = \{ \begin{pmatrix} v\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\\ \alpha\\ \beta\end{pmatrix} \mid \alpha, \beta \in \mathbb{R} \}, \tag{B.2}$$

where $v = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$ is a 2×2-matrix. The plane L can be written in the form (B.1) by writing

$$L = \{ \begin{pmatrix}0\\ 0\\ \alpha\\ \beta\end{pmatrix} + A_L \begin{pmatrix}0\\ 0\\ \alpha\\ \beta\end{pmatrix} \mid \alpha, \beta \in \mathbb{R} \}, \tag{B.3}$$

with the linear map A_L given by the 4×4-matrix $A_L = \left(\begin{smallmatrix} 0 & v \\ 0 & 0 \end{smallmatrix}\right)$. The bilinear form β_L on M_2 constructed from A_L is then given by

$$\begin{aligned} \beta_L(x, y) = \sigma(A_L(x), y) = \sigma(&\begin{pmatrix} ax_3 + bx_4 \\ cx_3 + dx_4 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ y_3 \\ y_4 \end{pmatrix}) \\ &= ax_3y_3 + bx_4y_3 + cx_3y_4 + dx_4y_4 \end{aligned}$$

for $x = (0, 0, x_3, x_4)^T$, $y = (0, 0, y_3, y_4)^T$. From now on we will identify 2×2-matrices A with bilinear forms on $\mathbb{R}^2$ by taking $A(x, y) = x^T Ay$. With this identification every plane L given by a matrix v and formula (B.2) can be identified with a 2×2-matrix $\beta_L = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. The Lagrange planes correspond precisely to the symmetric bilinear forms, so they corresponds to the symmetric matrices. If we put together the calculations we can conclude that the diffeomorphism L_1 : $\mathrm{Symm}^2(\mathbb{R}^2) \to \Lambda^0(E, \sigma, M_1)$ is given by the map

$$v = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \mapsto L_1(v) = \{ \begin{pmatrix} v\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ \alpha \\ \beta \end{pmatrix} \mid \alpha, \beta \in \mathbb{R} \} = \{ \begin{pmatrix} a\alpha + b\beta \\ b\alpha + c\beta \\ \alpha \\ \beta \end{pmatrix} \mid \alpha, \beta \in \mathbb{R} \}. \tag{B.4}$$

In a similar way (Λ^0, σ, M_2) is diffeomorphic to $\mathrm{Symm}^2(\mathbb{R}^2)$ by

$$v = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \mapsto L_2(v) = \{ \begin{pmatrix} \alpha \\ \beta \\ -v\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \end{pmatrix} \mid \alpha, \beta \in \mathbb{R} \} \in \Lambda^0(E, \sigma, M_2). \tag{B.5}$$

The other patches $\Lambda^0(E, \sigma, M_3)$ and $\Lambda^0(E, \sigma, M_3)$ can also be identified with $\mathrm{Symm}^2(\mathbb{R}^2)$ by

$$v = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \mapsto L_3(v) = \{ \begin{pmatrix} b\alpha + c\beta \\ \alpha \\ \beta \\ -a\alpha - b\beta \end{pmatrix} \mid \alpha, \beta \in \mathbb{R} \} \in \Lambda^0(E, \sigma, M_3),$$

$$v = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \mapsto L_4(v) = \{ \begin{pmatrix} \alpha \\ b\alpha + c\beta \\ -a\alpha - b\beta \\ \beta \end{pmatrix} \mid \alpha, \beta \in \mathbb{R} \} \in \Lambda^0(E, \sigma, M_4).$$

In summary: $\Lambda(E, \sigma)$ is covered by four standard coordinate patches $\Lambda^0(E, \sigma, M_j)$. Each of these patches is diffeomorphic to $\mathrm{Symm}^2(\mathbb{R}^2)$ by a map $L_j : \mathrm{Symm}^2(\mathbb{R}^2) \to \Lambda^0(E, \sigma, M_j)$.

The coordinate transformations between the different coordinate patches are needed in the main text. Using the diffeomorphisms L_j, it is not difficult to calculate these coordinate transformations. The conditions for which the transformations are valid are omitted.

$$\begin{aligned} \Lambda^0(E, \sigma, M_1) \to \Lambda^0(E, \sigma, M_2) : v = \begin{pmatrix} a & b \\ b & c \end{pmatrix} &\mapsto -v^{-1} = \frac{-1}{ac - b^2} \begin{pmatrix} c & -b \\ -b & a \end{pmatrix} \\ \Lambda^0(E, \sigma, M_1) \to \Lambda^0(E, \sigma, M_3) : v = \begin{pmatrix} a & b \\ b & c \end{pmatrix} &\mapsto \begin{pmatrix} -c^{-1} & bc^{-1} \\ bc^{-1} & -a + b^2c^{-1} \end{pmatrix} \\ \Lambda^0(E, \sigma, M_1) \to \Lambda^0(E, \sigma, M_4) : v = \begin{pmatrix} a & b \\ b & c \end{pmatrix} &\mapsto \begin{pmatrix} -a^{-1} & a^{-1}b \\ a^{-1}b & c - a^{-1}b^2 \end{pmatrix} \end{aligned} \tag{B.6}$$

B.3 The tangent space of a symplectic vector space

Suppose $L \in \Lambda(E, \sigma)$. If we choose a Lagrange plane M transversal to L then the neighborhood $\Lambda^0(E, \sigma, M)$ of L is diffeomorphic to $\mathrm{Symm}^2(L)$. The tangent space of $\Lambda(E, \sigma)$ at the point L is therefore diffeomorphic to the tangent space of $\mathrm{Symm}^2(L)$, which is just $\mathrm{Symm}^2(L)$. The identification of $\Lambda^0(E, \sigma, M)$ with $\mathrm{Symm}^2(L)$ depends on the choice of M. Surprisingly, the identification of the corresponding tangent spaces is independent of the choice of M.

Theorem B.3.1. *Let (E, σ) be a symplectic vector space and L, M transversal Lagrange planes in E. Denote by $\psi_{L,M} : \Lambda^0(E, \sigma, L) \to \mathrm{Symm}^2(L)$ the coordinate chart constructed in the proof of proposition B.1.2. The tangent map $T_L(\psi_{L,M})$ is independent of M and therefore yields a canonical identification of $T_L\Lambda(E, \sigma)$ with* $\mathrm{Symm}^2(L)$.

Proof. This follows directly from [10, Theorem 3.4.7]. □

Appendix C

Linear algebra

C.1 Bilinear forms

Let L be a finite-dimensional vector space. We denote by L^* the dual space of all linear forms on L. We define $\mathrm{Symm}^2(L)$ to be the space of all symmetric bilinear forms on L. For every linear map $A : L \to M$ there is a dual map $A^* : M^* \to L^*$ defined by $A^*(w)(x) = w(Ax)$ for $x \in L$ and $w \in M^*$.

Lemma C.1.1. *Let L be a finite-dimensional vector space. For every bilinear form $\beta \in \mathrm{Symm}^2(L)$ there is a unique linear map $B : L \to L^*$ such that for all $x, y \in L$ we have*

$$\beta(x, y) = (B(y))(x) = \langle x\,,\, By\rangle. \tag{C.1}$$

A symmetric β corresponds in this way precisely to a linear map B for which $B = B^*$. In this way we can identify symmetric bilinear forms on L with linear maps $L \to L^*$ that are equal to their transposed map.

Theorem C.1.2. *Let L be a finite-dimensional vector space. If $\gamma \in \mathrm{Symm}^2(L^*)$ then we define a linear form τ_γ on $\mathrm{Symm}^2(L)$ by*

$$\tau_\gamma : \beta \in \mathrm{Symm}^2(L) \to \mathrm{tr}[C \circ B], \tag{C.2}$$

where $B : L \to L^*$ and $C : L^* \to L$ are the maps corresponding to β and γ according to the identification in lemma (C.1.1). The map $\gamma \to \tau_\gamma$ is a canonical isomorphism of $\mathrm{Symm}^2(L^*)$ with $\mathrm{Symm}^2(L)^*$.

Proof. The maps C and B defined by lemma (C.1.1) are linear so their composition $C \circ B$ is a well-defined linear map. The map $\gamma \to \tau_\gamma$ is linear because $\gamma + \gamma' \to \tau_{\gamma+\gamma'}$ and

$$\begin{aligned}
\tau_{\gamma+\gamma'}(\beta) &= \mathrm{tr}[(C + C') \circ B] \\
&= \mathrm{tr}[C \circ B] + \mathrm{tr}[C' \circ B] \\
&= \tau_\gamma(\beta) + \tau_{\gamma'}(\beta), \tau_{\lambda\gamma} \qquad\qquad &= \mathrm{tr}[\lambda C \circ B] \\
&= \lambda\,\mathrm{tr}[C \circ B] = \lambda \tau_\gamma.
\end{aligned}$$

The map is also injective because if $\gamma \neq 0$ then $C \neq 0$ and therefore $\beta \to \mathrm{tr}[C \circ B]$ is unequal to zero. If $\dim L = n$ then $\dim L^* = n$ and $\dim(\mathrm{Symm}^2(L)^*) = \dim(\mathrm{Symm}^2(L^*)) = n(n+1)/2$. This implies that $\gamma \to \tau_\gamma$ is also surjective and therefore an isomorphism of vector spaces. □

Finally we want to give a description of the identification of $\mathrm{Symm}^2(L^*)$ with $\mathrm{Symm}^2(L)^*$ in local coordinates. Suppose $e_1, \ldots, e_n$ is a basis for L. Let $\omega_1, \ldots, \omega_n$ be the dual basis for L^*, defined by $\omega_i(e_j) = \delta_{ij}$. For $\mathrm{Symm}^2(L)$ we take the basis h_{ij}, $1 \leq i \leq j \leq n$ defined by

$$h_{ij}(e_k, e_l) = \begin{cases} 1 & \text{if } i = k, j = l \text{ or } i = l, j = k, \\ 0 & \text{otherwise.} \end{cases}$$

The dual basis in $\mathrm{Symm}^2(L)^*$ is denoted by η_{ij}, i.e. $\eta_{ij}(h_{i'j'}) = \delta_{ii'}\delta_{jj'}$. In the same way as for $\mathrm{Symm}^2(L)$ we can define a basis for $\mathrm{Symm}^2(L^*)$, which is denoted by k_{ij}.

In this notation we can write every symmetric bilinear form γ on L^* as $\gamma = \sum_{i,j} \gamma_{ij} k_{ij}$ with $\gamma_{ij} = \gamma(\omega_i, \omega_j)$. The corresponding map $C : L^* \to L$ was defined by formula (C.1), so if we write $C\omega_i = \sum_j C_{ij} e_j$ then it follows that $C_{ij} = \gamma_{ji}$. The element τ_γ acts on the basis h_{ij} of $\mathrm{Symm}^2(L)$ as

$$\begin{aligned}\tau_\gamma(h_{ij}) = \mathrm{tr}[C \circ H_{ij}] &= \sum_k \omega_k(C \circ H_{ij}(e_k)) \\ &= \sum_k \omega_k(C(\delta_{jk}\omega_i)) = \omega_j(C(\omega_i)) \\ &= \sum_k \omega_j(C_{ik} e_k) = C_{ij} = \gamma_{ji}.\end{aligned}$$

So in local coordinates $\gamma = \sum_{i,j} \gamma_{ij} k_{ij} \in \mathrm{Symm}^2(L^*)$ is mapped to $\tau_\gamma = \sum_{i,j} \gamma_{ji} \eta_{ij}$. Because γ is symmetric, we find that the identification of theorem C.1.2 is given in local coordinates by

$$\gamma = \sum_{i,j} \gamma_{ij} k_{ij} \to \tau_\gamma = \sum_{i,j} \gamma_{ij} \eta_{ij}. \tag{C.3}$$

C.2 Matrix calculations

If A is an $n \times n$-matrix we denote by $A(i,j)$ the matrix obtained from A by deleting the i-th row and j-th column. The matrix $A(i,j)$ is called a minor of A. We define the adjoint-matrix of A by $(A^{\mathrm{co}})_{ij} = (-1)^{i+j} \det A(j,i)$. Cramer's rule then says that $A^{\mathrm{co}} A = \det A$ (see [14, section 4.3] for a proof). If A is invertible we have a formula for the inverse $A^{-1} = (\det A)^{-1} A^{\mathrm{co}}$.

For the trace and the determinant of $n \times n$-matrices A and B we have

$$\begin{aligned}\det AB &= \det A \det B, \\ \mathrm{tr}\, AB &= \mathrm{tr}\, BA, \\ \mathrm{tr}\, A^T &= \mathrm{tr}\, A, \\ (AB)^{\mathrm{co}} &= B^{\mathrm{co}} A^{\mathrm{co}}.\end{aligned}$$

See [14] for a proof. For 2×2-matrices we have the following extra identity's

$$\begin{aligned}\det(A+B) &= \det A + \det B + \mathrm{tr}[AB^{\mathrm{co}}], \\ tr[AB] &= \mathrm{tr}\, A \,\mathrm{tr}\, B - \mathrm{tr}[A^{\mathrm{co}} B], \\ (A+B)^{\mathrm{co}} &= A^{\mathrm{co}} + B^{\mathrm{co}}, \\ \mathrm{tr}[A^{\mathrm{co}}] &= \mathrm{tr}\, A.\end{aligned}$$

All these identities follow by writing out the equations.

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