Hyperbolic Surfaces in the Grassmannian

P.T. Eendebak

Department of Mathematics, Utrecht University, Budapestlaan 6, 3584 CD, Utrecht, The Netherlands

Abstract

In this article we study real 2-dimensional surfaces in the Grassmannian of 2-planes in a 4-dimensional vector space. These surfaces occur naturally as the fibers of jet bundles of partial differential equations.

On the Grassmannian there is an invariant conformal quadratic form and we will use the structure induced by this quadratic form to study the surfaces. We give a topological classification of compact hyperbolic surfaces similar to the classification by Gluck and Warner (Duke Math J., **50**:1, 1983) of compact elliptic surfaces. In contrast with elliptic surfaces there are several topological possibilities for hyperbolic surfaces. We make a calculation of the differential invariants under the action of the group of conformal isometries. Finally, we analyze a class of surfaces called geometrically flat and show that within this class there exist many examples of non-trivial compact surfaces.

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Email address: `eendebak@math.uu.nl` (P.T. Eendebak).

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1 Introduction

In this article we study hyperbolic surfaces in the Grassmannian of 2-planes in a 4-dimensional vector space. This type of surface occurs naturally in the study of partial differential equations. See Section 3 for the relation between these surfaces and partial differential equations.

On the tangent space of the Grassmannian there is an invariant conformal quadratic form. The elliptic and hyperbolic surfaces correspond to the surfaces on which this conformal quadratic form restricts to a non-degenerate definite or indefinite conformal quadratic form, respectively. The elliptic surfaces have already been described by McKay [1,2] using complex numbers.

For the compact hyperbolic surfaces we obtain a topological classification (Theorem 11), similar to the classification for compact elliptic surfaces described by Gluck and Warner [3]. A compact hyperbolic surface is either a torus or a Klein bottle. We also study a special class of hyperbolic surfaces called the geometrically flat surfaces. We show that, even though the condition for a surface to be geometrically flat is quite rigid, there exist several different classes of geometrically flat compact surfaces. We conclude the study by giving a calculation of the local invariants of hyperbolic surfaces under the action of the general linear transformations of the vector space. Because the conformal isometry group is finite-dimensional, we can give in the generic case a complete description of the invariants at all orders.

2 Grassmannians

Let V be an n-dimensional vector space. The *Grassmannian* $\mathrm{Gr}_k(V)$ [4,5] is defined as the set of all k-dimensional linear subspaces of V. The k-dimensional linear subspaces of V are also called *k-planes* in V. The group $G = \mathrm{GL}(V)$ acts transitively on V and this induces a transitive action on $\mathrm{Gr}_k(V)$. The stabilizer group of a k-plane L is the group $H = \{ g \in \mathrm{GL}(V) \mid g(L) = L \}$. The Grassmannian is a homogeneous space G/H of dimension $k(n-k)$. There is a unique differentiable structure on $G/H = \mathrm{Gr}_k(V)$ such that $G \to G/H$ is a principal fiber bundle [6, Theorem 1.11.4].

We denote the manifold of oriented k-planes by $\widetilde{\mathrm{Gr}}_k(V)$. Locally $\mathrm{Gr}_k(V)$ and $\widetilde{\mathrm{Gr}}_k(V)$ are diffeomorphic. The space of oriented k-planes is a 2-fold cover of the space of unoriented k-planes.

Given an element $L \in \mathrm{Gr}_k(V)$ we can introduce local coordinates for $\mathrm{Gr}_k(V)$ in the following way. Select a complementary subspace M such that $L \oplus M = V$.

Let $\mathrm{Gr}_k^0(V, M)$ be the open subset of $\mathrm{Gr}_k(V)$ of k-planes that are transversal to M.

Lemma 1 *The space* $\mathrm{Lin}(L, M)$ *is diffeomorphic to* $\mathrm{Gr}_k^0(V, M)$ *through the map* $A \in \mathrm{Lin}(L, M) \mapsto \{ x + Ax \mid x \in L \} \in \mathrm{Gr}_k(V)$.

The diffeomorphisms described in the previous lemma for different k-planes L, M provide coordinate charts for $\mathrm{Gr}_k(V)$. The coordinate transformations between these charts are rational maps.

Let L_0 be a point in $\mathrm{Gr}_k(V)$. We define $\Sigma_{L_0} = \{ L \in \mathrm{Gr}_k(V) \mid L \cap L_0 \neq \{ 0 \} \}$. If we choose a transversal $(n-k)$-plane M and use the local coordinates from Lemma 1, then $\Sigma_{L_0} \cap \mathrm{Gr}_k^0(V, M) = \{ A \in \mathrm{Lin}(L_0, M) \mid \ker A \neq 0 \}$. If $n = 2k$, then Σ_{L_0} is determined by the $k \times k$-matrices with determinant zero. This is a hypersurface in the Grassmannian with a conical singularity at the zero matrix.

In the case of 2-planes in $V = \mathbb{R}^4$ there is another view of the Grassmannian. In the remainder of the paper we will assume V has dimension 4. Every 2-plane in V can be represented by 2 linearly independent vectors X, Y. Such a pair defines a non-zero element $X \wedge Y$ of $\Lambda^2(V)$. Since $\Lambda^4(V) \cong \mathbb{R}$ the map

$$\lambda : \Lambda^2(V) \to \Lambda^4(V) : \eta \mapsto \eta \wedge \eta$$

can be viewed as a homogeneous polynomial of degree 2. The elements $X \wedge Y$ that represent a 2-plane all satisfy $\lambda(X \wedge Y) = X \wedge Y \wedge X \wedge Y = 0$. Conversely, if an element $\eta \in \Lambda^2(V) \setminus \{ 0 \}$ satisfies $\lambda(\eta) = 0$, then it can be written as $\eta = X \wedge Y$ for two linearly independent vectors $X, Y \in V$.

Lemma 2 *The Grassmannian of* 2*-planes in a* 4*-dimensional vector space* V *is isomorphic to* $N = \{ \eta \in \Lambda^2(V) \mid \eta \neq 0, \lambda(\eta) = 0 \} / \mathbb{R}^* \subset \mathbb{P}(\Lambda^2(V))$.

The zero set of λ defines a smooth quadratic hypersurface in $\mathbb{P}(\Lambda^2(V))$. The description of the Grassmannian as a smooth quadratic is due to Plücker [7]. The oriented Grassmannian is isomorphic to the quadric defined by λ in $\Lambda^2(V)/\mathbb{R}^+$.

2.1 Conformal quadratic form

We recall the following well-known lemma [4, Lecture 16].

Lemma 3 *Let* L *be a* k*-plane in* $\mathrm{Gr}_k(\mathbb{R}^n)$. *Then* $T_L \mathrm{Gr}_k(\mathbb{R}^n)$ *is canonically isomorphic to* $\mathrm{Lin}(L, \mathbb{R}^n/L)$.

In the case that $n = 4$ and $k = 2$, we can identify $\mathrm{Lin}(L, V/L)$, after a choice of basis in L and V/L, with the space of 2×2-matrices. The determinant of a 2×2-matrix defines a quadratic form of signature $(2, 2)$. This gives a quadratic form on the tangent space of $\mathrm{Gr}_2(V)$ that depends on the choice of basis. Modulo a scalar factor this quadratic form is well-defined and hence we have defined a *conformal quadratic form* ξ on the tangent space of $\mathrm{Gr}_2(V)$ which is *invariant* with respect to the action of the group $\mathrm{GL}(V)$. For other introductions to this conformal quadratic form see [8, pp. 19–23] or [1, pp. 19–20].

The kernel of the action of $\mathrm{GL}(V)$ on $\mathrm{Gr}_2(V)$ is equal to the scalar multiples of the identity transformation. This implies $\mathbb{P}\,\mathrm{GL}(V)$ acts effectively on $\mathrm{Gr}_2(V)$. The action of both $\mathrm{GL}(V)$ and $\mathbb{P}\,\mathrm{GL}(V)$ on $\mathrm{Gr}_2(V)$ is by conformal transformations. This can be seen for example from the expression of this action in local coordinates, see formula (4) on page 7. The following lemma (proved in [9, Lemma 2.1.5]) completely characterizes the conformal isometries of the Grassmannian.

Lemma 4 *Let V be a real 4-dimensional vector space. The conformal isometry group of $\mathrm{Gr}_2(V)$ is equal to $\mathbb{P}\,\mathrm{GL}(V)$. The conformal isometry group of $\widetilde{\mathrm{Gr}}_2(V)$ is equal to the group $\mathbb{P}\,\mathrm{GL}^+(V)$ of orientation preserving projective linear transformations.*

Any conformal quadratic form ξ on a vector space W defines an *isotropic cone* $C = \{\, w \in W \otimes \mathbb{C} \mid \xi(w) = 0 \,\}$. If W is 2-dimensional and the conformal quadratic form is non-degenerate, then the isotropic cone consists of two distinct complex 1-dimensional linear subspaces which are called the *characteristic lines* of the conformal quadratic form. If the conformal quadratic form is definite, then the intersection of the isotropic cone with W consists of the origin. If the form is indefinite, then the intersection of the isotropic cone with W consists of two 1-dimensional lines in W. We call these lines the *characteristic lines* as well.

2.2 Plücker coordinates

We have described the Grassmannian $\mathrm{Gr}_2(V)$ as the space of elements η in $\Lambda^2(V)/\mathbb{R}^*$ that satisfy $\eta \wedge \eta = 0$. In this section we will use the eigenspaces of the Hodge $*$ operator to show that the Grassmannian $\mathrm{Gr}_2(V)$ is diffeomorphic to the direct product of two spheres.

Let e_1, e_2, e_3, e_4 form a basis for V. With respect to the volume form $\Omega = e_1 \wedge e_2 \wedge e_3 \wedge e_4$ we have the Hodge star operator $* : \Lambda^2(V) \to \Lambda^2(V)$. We define

$$\begin{aligned}
\alpha_1 &= (1/2)(e_1 \wedge e_2 + e_3 \wedge e_4), & \beta_1 &= (1/2)(e_1 \wedge e_2 - e_3 \wedge e_4),\\
\alpha_2 &= (1/2)(e_1 \wedge e_3 - e_2 \wedge e_4), & \beta_2 &= (1/2)(e_1 \wedge e_3 + e_2 \wedge e_4),\\
\alpha_3 &= (1/2)(e_1 \wedge e_4 + e_2 \wedge e_3). & \beta_3 &= (1/2)(e_1 \wedge e_4 - e_2 \wedge e_3).
\end{aligned}$$

The forms α_i, β_j satisfy

$$\alpha_i \wedge \beta_j = 0, \quad \alpha_i \wedge \alpha_j = \delta_{ij}\Omega, \quad \beta_i \wedge \beta_j = -\delta_{ij}\Omega.$$

The eigenspaces of the Hodge operator are $E_+ = \langle \alpha_j \rangle$, $E_- = \langle \beta_j \rangle$ corresponding to the eigenvalues 1 and -1 of $*$, respectively. We can decompose any $\eta \in \Lambda^2(V)$ in terms of these eigenspaces. Write $\eta = X^i\alpha_i + Y^j\beta_j$. The coefficients X^i, Y^j can be used to parameterize the Grassmannian and are called *Plücker coordinates*. The name Plücker coordinates is misleading because the coefficients do not define real coordinates for $\mathrm{Gr}_2(V)$. A pair (X, Y) only defines an element of the Grassmannian if the Plücker form λ is zero and two elements that are a scalar multiple of each other define the same element in the Grassmannian. The conformal quadratic form λ acts on η as

$$\lambda(\eta) = (X^1)^2 + (X^2)^2 + (X^3)^2 - (Y^1)^2 - (Y^2)^2 - (Y^3)^2.$$

Lemma 5 *Let S^+ and S^- be two copies of the 2-sphere $S^2 \subset \mathbb{R}^3$. Then the map*

$$S^+ \times S^- \to \Lambda^2(V)/\mathbb{R}^+ : (X, Y) \mapsto X^i\alpha_j + Y^j\beta_j$$

defines a diffeomorphism from $S^+ \times S^-$ to the oriented Grassmannian. The unoriented Grassmannian is diffeomorphic to $S^+ \times S^-/(-I, -I)$.

PROOF. This result is from [3]. Since $(X, Y) \in S^+ \times S^-$ satisfies $|X|^2 = |Y|^2 = 1$, the image of this map is contained in the kernel of the Plücker form λ.

2.3 Conformal group

In this section we analyze the action of the group of conformal transformations on the 2-planes in the tangent space of the Grassmannian.

The group $\mathrm{CO}(2, 2)$ of conformal transformations of $\mathbb{R}^4$ with a conformal quadratic form of signature $(2, 2)$ is isomorphic to $(\mathrm{GL}(2, \mathbb{R}) \times \mathrm{GL}(2, \mathbb{R}))/\mathbb{R}^* \cong \mathrm{SL}(2) \times \mathrm{SL}(2) \times H$, $H = \mathbb{R}^*$, see [8]. If we represent $\mathbb{R}^4$ by 2×2-matrices and the conformal quadratic form by the determinant, then the action of the conformal group is given by

$$(\tilde{\alpha}, \tilde{\delta}) \in \mathrm{CO}(2,2) : X \mapsto \tilde{\delta} X \tilde{\alpha}^{-1}. \tag{1}$$

Let $e_1, \ldots, e_4$ be the standard basis for $V = \mathbb{R}^4$ and take e_1, e_2 and e_3, e_4 as a basis for $L_0 = \mathbb{R}e_1 + \mathbb{R}e_2$ and $M = \mathbb{R}e_3 + \mathbb{R}e_4$, respectively. With these bases we can identify $\mathrm{Lin}(L_0, M) \cong \mathrm{Gr}_2^0(V, M)$ with the space of 2×2 matrices. The action of $g \in \mathrm{GL}(V)$ on the Grassmannian in these local coordinates is given by

$$g = \begin{pmatrix} \tilde{\alpha} & \tilde{\beta} \\ \tilde{\gamma} & \tilde{\delta} \end{pmatrix} : A \mapsto (\tilde{\gamma} + \tilde{\delta} A)(\tilde{\alpha} + \tilde{\beta} A)^{-1}. \tag{2}$$

Here $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$ and $\tilde{\delta}$ are 2×2-matrices. The action (2) might not be well-defined for all g since we are working in local coordinates for $\mathrm{Gr}_2^0(V, M)$, but it is well-defined for elements g near the identity and matrices A near the zero matrix.

The tangent space at a point L in the Grassmannian is given in these local coordinates by the space of 2×2-matrices as well and the conformal quadratic form ξ on $T\,\mathrm{Gr}_2(\mathbb{R}^4)$ is given by the determinant

$$\xi_L : T_L\,\mathrm{Gr}_2(\mathbb{R}^4) \to \mathbb{R} : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto AD - BC.$$

The point L_0 in the Grassmannian corresponds to the matrix $A = 0$. The stabilizer group H of L_0 is equal to the set of matrices

$$\begin{pmatrix} \tilde{\alpha} & \tilde{\beta} \\ 0 & \tilde{\delta} \end{pmatrix}, \tag{3}$$

with $\tilde{\alpha}$, $\tilde{\delta}$ invertible 2×2-matrices and $\tilde{b}$ an arbitrary 2×2-matrix.

We want to know how the stabilizer H acts on the tangent space of the Grassmannian. Suppose that $t \mapsto tX$ is a curve through the point L_0 that represents a tangent vector in the Grassmannian. The group H acts on this curve as

$$t \mapsto \tilde{\delta} t X (\tilde{\alpha} + \tilde{\beta} t X)^{-1} = t \tilde{\delta} X \tilde{\alpha}^{-1} + \mathcal{O}(t^2).$$

So the action of H on the 2-planes in the tangent space is given by

$$\begin{pmatrix} \tilde{\alpha} & \tilde{\beta} \\ 0 & \tilde{\delta} \end{pmatrix} \cdot X = \tilde{\delta} X \tilde{\alpha}^{-1}. \tag{4}$$

This is precisely the action (1) of the conformal group mentioned above.

A 2-dimensional linear subspace of the tangent space $T\,\mathrm{Gr}_2(V)$ will be called a *tangent* 2*-plane* or just a tangent plane.

Definition 6 *A tangent* 2*-plane in* $T\,\mathrm{Gr}_2(V)$ *is called* elliptic *if the conformal quadratic form* ξ *restricts to a definite non-degenerate quadratic form. A tangent 2-plane in* $T\,\mathrm{Gr}_2(V)$ *is called* hyperbolic *if the conformal quadratic form restricts to a non-degenerate quadratic form of signature* $(1, 1)$*.*

The action of the conformal group on the 2-planes in the tangent space of $\mathrm{Gr}_2(V)$ at a point has exactly 5 orbits of which 2 are open. The two open orbits correspond to the elliptic and hyperbolic 2-planes. Representative elements for the 5 orbits are given in [10, Section 7.1, Case 2, p. 272].

Theorem 7 ([9, Theorem 2.1.7]) *The general linear group acts transitively on the Grassmannian of* 2*-planes. At each point in the Grassmannian the stabilizer subgroup of that point acts transitively on the elliptic tangent planes and also transitively on the hyperbolic tangent planes.*

2.4 Product structures

An (almost) product structure is the analogue of an (almost) complex structure.

Definition 8 (Product structure) *Let* V *be a* $2n$*-dimensional real vector space. A* product structure *(complex structure) on* V *is an endomorphism* $K : V \to V$ *such that* $K^2 = \mathrm{I}$ *(*$K^2 = -\mathrm{I}$*) and the eigenvalues* ± 1 *(*$\pm i$*) of* K *both occur with geometric multiplicity* n*.*

The condition that $K^2 = \mathrm{I}$ implies that the eigenspaces $V_\pm$ of K for the eigenvalues ± 1 span V. Hence the algebraic multiplicity will always equal the geometric multiplicity.

Given a product structure K we define $V_\pm$ as the eigenspace of K for eigenvalue ± 1. Then from the definition it follows that $V = V_+ \oplus V_-$. Conversely, if we have a direct sum $V = V_+ \oplus V_-$, then we can define a product structure on V by $K|_{V_\pm} = \pm\mathrm{I}$. Note that for a complex structure J on V, the complexified vector space is the direct sum $V \otimes \mathbb{C} = V_+ \oplus V_-$ of the eigenspaces of J

corresponding to $\pm i$. A product structure on an even-dimensional vector space does not determine an orientation. This is in contrast with a complex structure that does determine an orientation.

3 Partial differential equations

One application of surfaces in the Grassmannian is the occurrence of these surfaces in the geometric treatment of partial differential equations.

Consider a first order system of partial differential equations in two unknown functions u, v and two variables x, y. The system is *determined* if it is given by two equations

$$F(x, y, u, v, u_x, u_y, v_x, v_y) = 0, \quad G(x, y, u, v, u_x, u_y, v_x, v_y) = 0, \tag{5}$$

such that the matrix

$$\begin{pmatrix} F_{u_x} & F_{u_y} & F_{v_x} & F_{v_y} \\ G_{u_x} & G_{u_y} & G_{v_x} & G_{v_y} \end{pmatrix}$$

has rank 2 at all points $F = G = 0$. Let B equal $\mathbb{R}^2 \times \mathbb{R}^2$ with coordinates x, y, u, v. The first order system (5) defines for each point $b = (x, y, u, v)$ in B a codimension 2 surface in $\mathrm{Gr}_2(T_b B)$. This defines a codimension 2 submanifold of $\mathrm{Gr}_2(TB)$ called the *equation manifold* of the system (5). The tangent space to the graph of pair of functions $u(x, y), v(x, y)$ at the point (x, y) is a 2-dimensional linear subspace of $V = T_{(x,y,u,v)}B$ and hence an element of $\mathrm{Gr}_2(T_b B)$. A pair of functions $u(x, y), v(x, y)$ is a solution to the system of partial differential equations (5) if and only if the tangent spaces to the graph of this pair are all contained in the equation manifold of the system. The surfaces in $\mathrm{Gr}_2(T_b B)$ for $b \in B$ defined by the system are elliptic or hyperbolic in the sense to be described in Section 2 below, if the first order system is elliptic or hyperbolic in the classical sense, see [9, Remark 4.6.2]. We should note that any hyperbolic surface can be realized locally as one of the fibers of a system of partial differential equations.

The point transformations of the systems mentioned above with a fixed point at $b \in B$ act on V by linear transformations. Hence these transformations induce actions on the Grassmannian by conformal transformations. The invariants of hyperbolic surfaces and the topological type of compact hyperbolic surfaces (both explained in the sections below) are therefore invariants of these particular systems of partial differential equations under point transformations.

Also for other types of system of partial differential equations, such as second order scalar equations in the plane [9], these surfaces occur. For second order scalar equations the invariants of hyperbolic surfaces are invariants for the contact transformations.

4 Hyperbolic surfaces in the Grassmannian

Let S be a surface in the Grassmannian. The conformal quadratic form on the tangent space of the Grassmannian restricts to a conformal quadratic form on the tangent space of S. For generic tangent spaces the form is non-degenerate and is either definite (elliptic tangent planes) or indefinite (hyperbolic tangent planes). If the conformal quadratic form is definite this defines an almost complex structure on the surface and if the form is indefinite this defines an almost product structure on the surface. The surfaces with an almost complex structure or almost product structure are always integrable and have no local invariants. So studying the local geometry of these surfaces with the additional conformal structure itself is not very interesting.

However, the surfaces are embedded in the Grassmannian and it is very interesting to study the surfaces in the Grassmannian under the conformal isometry group of the Grassmannian. The reason for this is that, as explained in the previous section, the point and contact transformations of systems of partial differential equations often induce actions on Grassmannians by conformal transformations. Here we write down the theory of surfaces to which the conformal quadratic form restricts to an indefinite quadratic form (the hyperbolic case). Part of the elliptic case was already done by McKay [1, Chapter 4].

Let S be a surface in $\mathrm{Gr}_2(V)$. At each point $s \in S$ the tangent space T_sS has dimension 2 and the conformal quadratic form restricts to a conformal quadratic form on T_sS. We call the point s *elliptic* or *hyperbolic* if the tangent space of S at s is an elliptic or hyperbolic tangent plane, respectively. A surface for which all points are elliptic or hyperbolic is called an *elliptic surface* [11,1] or *hyperbolic surface*, respectively [2].

For a hyperbolic surface the conformal quadratic form restricts on the tangent space of the surface to a non-degenerate conformal quadratic form of signature $(1, 1)$. The kernel of the quadratic form is given by two lines in the tangent space. The vectors in the two lines are called the *characteristic vectors*. Since these characteristic vectors depend smoothly on the point of the surface, the characteristic vectors locally define a pair of transversal rank 1 distributions.

[2] The reader should not confuse the term hyperbolic surface with the surfaces of constant negative curvature.

The integral curves of these distributions are called the *characteristic curves.*

4.1 Standard hyperbolic tori

Every product structure on a 4-dimensional vector space V determines a hyperbolic surface. Let K be a product structure on V. Then we define the *standard hyperbolic torus* $\mathrm{Gr}_2(V, K)$ associated to K as the set of all 2-planes that are K-invariant and satisfy the non-degeneracy condition that K restricted to the 2-plane is not equal to $\pm I$. The elements of $\mathrm{Gr}_2(V, K)$ are called *hyperbolic lines* This definition can be compared to the definition of the *complex lines* for a complex structure in [1, p. 14]. The standard hyperbolic torus defined by a product structure is topologically indeed a torus. If we let $V_\pm \subset V$ be the eigenspaces of the product structure K, then $\mathrm{Gr}_1(V_+) \times \mathrm{Gr}_1(V_-) \to \mathrm{Gr}_2(V, K) : (l_1, l_2) \mapsto l_1 + l_2$ is an isomorphism.

4.2 Intersection curves

In this section we will analyze the intersection of a hyperbolic surface S with Σ_{L_0} (see page 3) for L_0 a point on the hyperbolic surface. The manifold Σ_{L_0} has dimension 3 and has a singularity at L_0. We will prove that locally the intersection of S and Σ_{L_0} looks like two curves intersecting transversally at L_0.

First we introduce local coordinates around the point L_0 in the Grassmannian. Let L_0 be the plane in $\mathrm{Gr}_2(\mathbb{R}^4)$ spanned by the two vectors $(1, 0, 0, 0)^{\mathrm{T}}$ and $(0, 1, 0, 0)^{\mathrm{T}}$ and let $M = \mathbb{R}(0, 0, 1, 0)^{\mathrm{T}} + \mathbb{R}(0, 0, 0, 1)^{\mathrm{T}}$. We use the local coordinates for the Grassmannian described on page 6. In these local coordinates we have

$$L_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \Sigma_{L_0} \cap \mathrm{Gr}_2^0(V, M) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 0 \right\}.$$

The surface S is then given as a 2-dimensional surface in the space of 2×2-matrices and the point L_0 corresponds to the zero matrix. Since the general linear group acts transitively on the hyperbolic tangent planes, we can arrange by a coordinate transformation that the tangent space to S is spanned at the point L_0 by the two tangent vectors

$$X_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

In these coordinates we can parameterize the surface S using two coordinates a, b as

$$\Gamma : U \subset \mathbb{R}^2 \to S : (a,b) \mapsto \begin{pmatrix} a & \phi(a,b) \\ \psi(a,b) & b \end{pmatrix},$$

with ϕ and ψ functions that vanish up to first order in a, b.

The manifold Σ_{L_0} is given by the 2-planes that have non-trivial intersection with L_0. These planes are precisely the planes for which the corresponding 2×2-matrices in local coordinates have zero determinant. Then $S \cap \Sigma_{L_0}$ is given by the condition $ab - \phi(a,b)\psi(a,b) = 0$. But the product $\phi(a,b)\psi(a,b)$ is at least of order 4 in a and b, hence by the Morse lemma this set looks locally like the zero set of ab which is a cross at the origin. We call the two curves the *intersection curves* of the surface S through the point L_0.

Example 9 (Standard hyperbolic torus) *The standard product structure K on $\mathbb{R}^4$ is given by the direct product $\mathbb{R}^2 \times \mathbb{R}^2$. Let $S = \mathrm{Gr}_2(\mathbb{R}^4, K)$ be the surface of hyperbolic lines in $\mathbb{R}^4$ for this product structure. The surface can be parameterized as*

$$(\theta, \phi) \mapsto \mathbb{R}\,(\cos\theta, 0, \sin\theta, 0)^{\mathrm{T}} + \mathbb{R}\,(0, \cos\phi, 0, \sin\phi)^{\mathrm{T}} \in \mathrm{Gr}_2(\mathbb{R}^4).$$

In the local coordinates introduced previously we have

$$S \cap \mathrm{Gr}_2^0(V, M) = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mid a, d \in \mathbb{R} \right\}.$$

The intersection of S and Σ_{L_0} is easy to calculate and is given by

$$S \cap \Sigma_{L_0} \cap \mathrm{Gr}_2^0(V, M) = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{R} \right\} \cup \left\{ \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} \mid d \in \mathbb{R} \right\}.$$

For every point L on the surface the characteristic curves through L are equal to the intersection curves through L defined by $\Sigma_L \cap S$.

We have proved that for a general hyperbolic surface S and point L_0 on this surface the intersection $\Sigma_{L_0} \cap S$ looks locally like two curves intersecting transversally at L_0. We can compare this pair of curves with the characteristic curves through the same point L_0. In general the characteristic curves and the intersection curves through a point L_0 are different, although at the point L_0 they have at least contact of order 2 [9, p. 65]. It can also happen that the characteristic curves through L_0 and the curves determined by $\Sigma_{L_0} \cap S$ are identical (see Example 9 above and Section 4.4 on geometrically flat surfaces).

Example 10 *We consider the surface defined in local coordinates for the Grassmannian by the matrices*

$$\Gamma(a,b) = \begin{pmatrix} a & a^2 \\ a^2 & b \end{pmatrix}.$$

The intersection curves through the origin follow from the equation $\det(\Gamma(a,b) - \Gamma(0,0)) = ab - a^4 = a(b - a^3) = 0$. *Here we can explicitly factorize the equation and this gives the intersection curves* $a = 0$ *and* $b = a^3$. *The characteristic lines at a point* (a,b) *are spanned by*

$$\begin{pmatrix} 1 & 2a \\ 2a & 4a^2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Integrating the characteristic lines defined by the first matrix yields the characteristic curves $(a(t), b(t)) = (a_0 + t, b_0 + (4/3)((a_0 + t)^3 - (a_0)^3))$. *Integration of the other matrix gives* $(a(t), b(t)) = (a_0, b_0 + t)$. *We immediately see that the intersection curves* $a =$ *constant overlap with the characteristic curves, but the other intersection curves do not overlap with the characteristic curves.*

4.3 Compact hyperbolic surfaces

Gluck and Warner [3] proved that every connected compact elliptic surface in the oriented Grassmannian can be deformed through elliptic surfaces to a Riemann sphere given by the complex lines for a complex structure on V. For the hyperbolic surfaces the situation is more complicated. A connected compact hyperbolic surface can be topologically a torus or Klein bottle and these different types of surfaces can never be deformed into each other.

Theorem 11 *Let* S *be a connected compact hyperbolic surface in* $\mathrm{Gr}_2(V)$ *or* $\widetilde{\mathrm{Gr}}_2(V)$. *Then* S *is either a* torus *or a* Klein bottle.

PROOF. The condition that a surface in the Grassmannian is hyperbolic implies that at each point of the surface there are two characteristic lines in the tangent space. Locally, we can always choose a basis of the tangent space to S consisting of two non-zero vector fields tangent to these characteristic lines. We can locally make the choice of such a basis unique by choosing a metric on the surface, an order for the two characteristic lines (so we label one of the characteristics as the first and the other as the second characteristic line) and a positive direction for each of the characteristic lines.

We can also choose a global metric for the surface (for example the metric induced from the diffeomorphism of the Grassmannian to $S^+ \times S^-$), but it is not always possible to make a global choice of order of the characteristics and directions. We can always pass to a cover of the surface on which the basis of vector fields is globally defined. We need at most a $8:1$ cover for this. First a $2:1$ cover for the ordering of the characteristic lines and then two times a $2:1$ cover for the direction of each of the characteristic lines.

Next consider the case of a compact hyperbolic surface. The covering surface is also compact and it is orientable. The covering surface has a trivial tangent space and this implies the surface has Euler characteristic zero; topologically the surface is a torus. Since the covering surface has Euler characteristic zero, the original surface is a compact surface with Euler characteristic zero and must be either a torus or a Klein bottle. The original surface is a torus if it is orientable and a Klein bottle if it is non-orientable.

There exist explicit examples of compact surfaces in both the oriented and unoriented Grassmannian that are diffeomorphic to a Klein bottle, see the examples below. The standard hyperbolic torus defined by a product structure is a compact hyperbolic surface that is homeomorphic to a torus.

Gluck and Warner proved not only that every compact elliptic surface is a 2-sphere, but even that every compact elliptic surface can be deformed to the standard elliptic surface. Again for compact hyperbolic surfaces the situation is more complicated. The oriented and unoriented Grassmannian are both connected, but the oriented Grassmannian (which is the product of two spheres) is simply connected and hence the fundamental group is trivial. The unoriented Grassmannian has fundamental group $\pi_1(\mathrm{Gr}_2(V)) \simeq \mathbb{Z}/2\mathbb{Z}$. There exist compact hyperbolic surfaces homeomorphic to a torus in the unoriented Grassmannian for which one of the generators of the fundamental group of the torus defines a non-trivial element in the fundamental group of the unoriented Grassmannian, but also compact hyperbolic surfaces for which the generators are all trivial in the fundamental group of the unoriented Grassmannian. Examples of both types are given in Example 14. These different surfaces can not be deformed into each other.

The author is not aware of any other topological obstructions against deformations, besides the topology of the surface and the mapping on fundamental groups. The examples in this section show these invariants are not enough to give a complete classification of the compact hyperbolic surfaces up to isotopy.

Example 12 *We consider the oriented Grassmannian $\widetilde{\mathrm{Gr}}_2(V)$ as the product of two spheres $S^+ \times S^-$. A family of immersed surfaces in the Grassmannian* *is given by*

$$\Phi : (s,t) \mapsto \begin{pmatrix} \cos(s) \\ 0 \\ \sin(s) \end{pmatrix} \times \begin{pmatrix} \cos(t) \\ \sin(\alpha s)\sin(t) \\ \cos(\alpha s)\sin(t) \end{pmatrix}.$$

The tangent space at a point of the surface is spanned by the two vectors $\Phi_s = \partial\Phi/\partial s$ *and* $\Phi_t = \partial\Phi/\partial t$. *Solving the characteristic equation* $\xi(a\Phi_s + b\Phi_t) = 0$, *where* ξ *is the conformal quadratic from on the tangent space of the Grassmannian, yields* $b = \pm a\sqrt{1 - \alpha^2 + \alpha^2 \cos^2(s)}$. *For* $|\alpha| < 1$ *the surface has two distinct real characteristics at each point and hence the surface is hyperbolic.*

For $\alpha = 0$ *we have an embedded torus. The standard torus* $T = R/(2\pi\mathbb{Z}) \times R/(2\pi\mathbb{Z})$ *is embedded as the product of two great circles; the explicit parameterization is given by*

$$T \to S^+ \times S^- : (s,t) \mapsto \left((\cos(s), 0, \sin(s))^{\mathrm{T}}, (\cos(t), 0, \sin(t))^{\mathrm{T}}\right).$$

For $\alpha = 1/2$ *the surface is a globally defined and compact surface* K*; topologically the surface is a Klein bottle. A* $2:1$ *cover of the torus* $\tilde{T} = R/(4\pi\mathbb{Z}) \times R/(2\pi\mathbb{Z})$ *to the Klein bottle* $K \subset S^+ \times S^-$ *is*

$$\tilde{T} \to S^+ \times S^- : (s,t) \mapsto \left(\begin{pmatrix} \cos(s) \\ 0 \\ \sin(s) \end{pmatrix}, \begin{pmatrix} \cos(t) \\ \sin(s/2)\sin(t) \\ \cos(s/2)\sin(t) \end{pmatrix} \right).$$

Example 13 *Let* $\gamma : \mathbb{R}/2\pi\mathbb{Z} \to S^+ \subset \mathbb{R}^3$ *be an embedding of the circle into the 2-sphere with the properties* $\gamma(s+\pi) = -\gamma(s)$ *for all* s *and* $|\gamma'(s)| > 1$ *for all* s. *Such embeddings are easy to construct by taking deformations of great circles and then reparameterizing by arc length. We define* $T = R/(2\pi\mathbb{Z}) \times R/(2\pi\mathbb{Z})$ *and* $\Phi : T \to \widetilde{\mathrm{Gr}}_2(V) : (s,t) \mapsto \left(\gamma(s), (\cos s \cos t, \sin s \cos t, \sin t)^{\mathrm{T}}\right)$. *The conformal quadratic form on the tangent space takes the form*

$$\xi(a\Phi_s + b\Phi_t) = a^2(|\gamma'(s)|^2 - \cos^2(t)) - b^2.$$

This is an indefinite non-degenerate quadratic form in a, b *at all points. Hence the surface defined by* Φ *is a hyperbolic surface. The image of the torus* T *is a torus in the oriented Grassmannian. The projection of* $\widetilde{\mathrm{Gr}}_2(V)$ *to* $\mathrm{Gr}_2(V)$ *induces a* $2:1$ *cover of the torus over a Klein bottle in the unoriented Grassmannian.*

Example 14 (Compact surfaces and the fundamental group) *Let* T *be the torus* $R/(2\pi\mathbb{Z}) \times R/(2\pi\mathbb{Z})$ *and let* z *be a constant with* $0 < z < 1$. *We* *define two compact hyperbolic surfaces in* $\mathrm{Gr}_2(V) = (S^+ \times S^-)/(-I,-I)$ *by*

$$\Phi^1 : T \to \mathrm{Gr}_2(V) : (s,t) \mapsto \left(\begin{pmatrix} \sqrt{1-z^2}\cos(s) \\ \sqrt{1-z^2}\sin(s) \\ z \end{pmatrix}, \begin{pmatrix} \cos(t) \\ \sin(t) \\ 0 \end{pmatrix} \right),$$

$$\Phi^2 : T \to \mathrm{Gr}_2(V) : (s,t) \mapsto \left(\begin{pmatrix} \cos(s/2) \\ \sin(s/2) \\ 0 \end{pmatrix}, \begin{pmatrix} \cos(t+s/2) \\ \sin(t+s/2) \\ 0 \end{pmatrix} \right).$$

Both maps Φ^1, Φ^2 *are embeddings of the torus* T *into the unoriented Grassmannian.*

Let γ *be the curve in* T *defined by* $s \mapsto (s,0)$*. Then* γ *defines a non-trivial element* $[\gamma]$ *in the fundamental group of* T*. The embedding* Φ^j *induces a homomorphism* Φ^j_* *from the fundamental group* $\pi_1(T)$ *to* $\pi_1(\mathrm{Gr}_2(V))$*. The image* $(\Phi^1)_*([\gamma])$ *is trivial in* $\pi_1(\mathrm{Gr}_2(V))$*, the image* $(\Phi^2)_*([\gamma])$ *is non-trivial in* $\pi_1(\mathrm{Gr}_2(V))$*.*

4.4 Geometrically flat surfaces

We define a hyperbolic surface to be *geometrically flat* if the characteristic curves and the intersection curves are identical. From Example 9 it is clear that the standard hyperbolic tori are geometrically flat. The converse is not true. The space of all standard hyperbolic tori in a Grassmannian is finite-dimensional. But the surfaces in Example 16 and Example 20 show that geometrically flat surfaces can depend on an arbitrary function and hence the space of geometrically flat surfaces is not finite-dimensional. So not all these surfaces can be standard hyperbolic tori and this proves the class of all geometrically flat surfaces is much larger then the class of all standard hyperbolic tori.

To analyze the structure of geometrically flat surfaces we start with an elementary lemma.

Lemma 15 *Let* $V = \mathbb{R}^4$ *and let* L_1, L_2, L_3 *be 2-dimensional linear subspaces such that* $\dim L_1 \cap L_2 = \dim L_1 \cap L_3 = \dim L_2 \cap L_3 = 1$*. Then the* L_j *are all contained in a 3-dimensional linear subspace* $L = L_1 + L_2 + L_3$ *or the three subspaces have a 1-dimensional linear subspace* $l = L_1 \cap L_2 \cap L_3$ *in common, or both.*

PROOF. Assume that $L_1 \cap L_2 \cap L_3 = \{ 0 \}$, so the subspaces have no line in common. Pick vectors e_1, e_2, e_3 in V such that $L_1 \cap L_2 = \mathbb{R}e_1$, $L_1 \cap L_3 = \mathbb{R}e_2$ and $L_2 \cap L_3 = \mathbb{R}e_3$. We cannot have $\mathbb{R}e_1 = \mathbb{R}e_2$ since this would imply that $\mathbb{R}e_1 \subset L_1 \cap L_2 \cap L_3$. Hence $L_1 = \mathbb{R}e_1 + \mathbb{R}e_2$. Since $\{ 0 \} = L_1 \cap L_2 \cap L_3 = L_1 \cap \mathbb{R}e_3$ we see that e_3 is not in the span of e_1, e_2. Hence the vectors e_1, e_2, e_3 are linearly independent. From the construction of e_1, e_2, e_3 it is clear that $L_1 + L_2 + L_3 = \mathbb{R}e_1 + \mathbb{R}e_2 + \mathbb{R}e_3$ and that $\dim(L_1 + L_2 + L_3) = 3$.

Let S be a geometrically flat surface in $\mathrm{Gr}_2(V)$. Let L_1, L_2, L_3 be three different points on the same characteristic curve γ. Since the surface is geometrically flat, one of the intersection curves through the point L_1 must be identical to the characteristic curve γ. Therefore both L_2 and L_3 must have non-zero intersection with L_1 and for the same reason L_2 and L_3 must have non-zero intersection. Recall that the points L_k are elements of the Grassmannian and hence 2-dimensional linear subspaces of V. Because the points L_1, L_2 and L_3 are different points, the intersections must be 1-dimensional and we can apply Lemma 15. This leads to the conclusion that locally there are three types of characteristic curves γ on a geometrically flat surface.

1) All points L on γ have a line l_1 in common and are contained in a three-dimensional subspace l_3.
2) All points L on γ have a line l_1 in common. The points L are not contained in a subspace of dimension three.
3) All points L on γ are contained in a three-dimensional subspace l_3. The points on γ do not have a line in common.

We say a characteristic curve is of type $(2')$ if the characteristic curve is either of type (1) or of type (2). We say a characteristic curve is of type $(3')$ if the characteristic curve is either of type (1) or of type (3). For a hyperbolic surface the type of the characteristic curves does not need to be constant. An example of such a surface is given in Example 16.

Let $\Gamma : (a, b) \mapsto \Gamma(a, b) \in \mathrm{Gr}_2(V)$ be a hyperbolic surface such that the characteristic curves are given by the equations $a = \text{constant}$ and $b = \text{constant}$. Whenever we have a hyperbolic surface parameterized in this way, we will call the curves defined by $b = \text{constant}$ the *horizontal characteristic curves* and the curves $a = \text{constant}$ the *vertical characteristic curves*. For a surface with (locally) constant type there are nine possibilities: the horizontal characteristic curves can have type (1), (2) or (3) and the vertical characteristic curves as well. If we allow to switch the characteristic curves, then there are only six types. We will say that a geometrically flat surface is of type (i, j) if the horizontal characteristic lines are of type (i) and the vertical characteristics are of type (j).

Example 16 (Changing type) *Let $P_{x_0,x_1}(x)$ be a smooth bump function that is zero outside the region $x_0 < x < x_1$ and non-zero inside this region. We then define $\phi_1(a) = P_{0,1}(a)$, $\phi_2(a) = P_{2,3}(a)$, $\psi_1(b) = P_{0,1}(b)$, $\psi_2(b) = P_{2,3}(b)$. Let S be the surface given in local coordinates by the embedding*

$$(a,b) \mapsto \Gamma(a,b) = \begin{pmatrix} a & \phi_1(a)\psi_1(b) \\ \phi_2(a)\psi_2(b) & b \end{pmatrix}.$$

The embedding Γ defines a hyperbolic surface and at each point (a,b) the matrices $\partial\Gamma/\partial a$ and $\partial\Gamma/\partial b$ are singular. This means that the characteristic curves are given by the lines $a =$ constant and $b =$ constant. To show that the intersection curves coincide with the characteristic curves consider an arbitrary point (a,b). The point $\Gamma(\tilde{a},\tilde{b})$ is contained in $\Sigma_{\Gamma(a,b)}$ if and only if $\det(\Gamma(a,b) - \Gamma(\tilde{a},\tilde{b})) = 0$. *Consider the points $(\tilde{a},b)$. For these points we have*

$$\det(\Gamma(a,b) - \Gamma(\tilde{a},b)) = (\phi_1(a) - \phi_1(\tilde{a}))\,(\phi_2(a) - \phi_2(\tilde{a}))\,\psi_1(b)\psi_2(b).$$

Since $\psi_1(b)\psi_2(b)$ is identically zero this shows that all points $\Gamma(\tilde{a},b)$ are in $\Sigma_{\Gamma(a,b)}$. This proves that the characteristic curves $b =$ constant coincide with the intersection curves. A similar analysis shows that also the lines $a =$ constant coincide the with intersection curves.

The hyperbolic surface in this example has changing type of characteristics. In Figure 1 the different regions on the surface are separated by black lines and the types are indicated. For example in the region $1 \leq a \leq 2$, $1 \leq b \leq 2$ the surface has type (2, 3). The points on the horizontal characteristic curve $b =$ constant are 2-planes that all have the line spanned by the vector $(0,1,0,b)^{\mathrm{T}}$ in common. The points on the vertical characteristic curve $a =$ constant are all 2-planes in the 3-dimensional subspace spanned by the vectors $(1,0,a,0)^{\mathrm{T}}$, $(0,1,0,0)^{\mathrm{T}}$ and $(0,0,0,1)^{\mathrm{T}}$. This single example shows that all possible combinations of type (i,j) exist for hyperbolic surfaces.

Example 17 (Geometrically flat surface of type (2′, 2′)) *Let γ and δ be two curves in $\mathrm{Gr}_1(\mathbb{R}^4)$ and define $\Gamma(s,t) = \gamma(s) + \delta(t)$. Assume that $\gamma(0) \neq \delta(0)$ and the tangent map of Γ at $(0,0)$ is injective. Then Γ (locally near $L_0 = \Gamma(0,0)$) defines a surface S in $\mathrm{Gr}_2(\mathbb{R}^4)$. If the tangent plane $T_{L_0}S$ to the surface at L_0 is a hyperbolic tangent plane, then S is a hyperbolic surface near L_0.*

This surface has the property that every point $\Gamma(s,t_0)$ on the curve $\phi_{t_0} : s \mapsto \Gamma(s,t_0)$ contains the line $\delta(t_0)$. Hence the intersection curves through the points L on this curve are all tangent to the curve ϕ_{t_0}. Since the intersection curves are always tangent to the characteristic curves this proofs that ϕ_{t_0} is a characteristic curve for the surface. In a similar way it follows that the curves $\psi_{s_0} : t \mapsto \Gamma(s_0,t)$ are characteristic curves and intersection curves for the

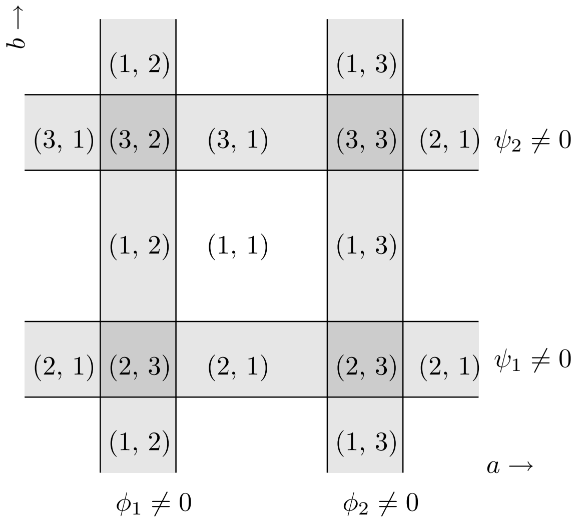

Figure 1. Geometrically flat surface with changing type of curves

points on ψ_{s_0}.

This surface is geometrically flat and the type is (2′,2′) because the points on the characteristic line ϕ_{t_0} *have the 1-dimensional linear subspace* $\delta(t_0)$ *in common and the points on the characteristic line* ψ_{s_0} *have the 1-dimensional linear subspace* $\gamma(s_0)$ *in common.*

Type (2′, 3′). Let S be a geometrically flat hyperbolic surface in $\mathrm{Gr}_2(V)$ of type $(2', 3')$ given by $(a, b) \mapsto \Gamma(a, b)$. For every point $\Gamma(a, b) \in S$ the points on the horizontal characteristic curve (which is of type $(2')$) through $\Gamma(a, b)$ have a line $l_1(b)$ in common. The points on the vertical characteristic curve through $\Gamma(a, b)$ are all contained in a 3-dimensional subspace $l_3(a)$. The lines $l_1(b)$ and the 3-dimensional spaces $l_3(a)$ satisfy the relation $l_1(b) \subset \Gamma(a, b) \subset l_3(a)$. This relation implies that

$$\bigcup_b l_1(b) \subset \bigcap_a l_3(a).$$

We use the notation $\sum_b l_1(b)$ to denote the span of the elements in $\bigcup_b l_1(b)$. Then it is clear that $\sum_b l_1(b)$ is a linear subspace of $\bigcap_a l_3(a)$.

The lines $l_1(b)$ and the 3-dimensional subspaces $l_3(a)$ must both vary as we vary a and b. For example if $l_1(b)$ is constant near $L_0 = \Gamma(a_0, b_0)$, then near L_0 all points on the surface have a single line $l_1 = l_1(b_0)$ in common. But then near L_0 the intersection $\Sigma_{\Gamma(a_0,b_0)} \cap S$ is equal to S and this is not possible. This implies that there is a unique 2-dimensional linear subspace L such that

$$\sum_b l_1(b) = L = \bigcap_a l_3(a). \tag{6}$$

If we assume the surface is connected, then the special point L is not a point on the surface.

Example 18 (Compact surfaces of type (2′, 3′)) *In this example we will make a construction of a large class of compact hyperbolic surfaces of class (2′, 3′). Recall that for any surface of type (2′, 3′) there is a unique* 2*-plane L that satisfies the equation* (6)*. We define*

$$F_L = \{ (l_1, l_2, l_3) \in \mathrm{Gr}_1(V) \times \mathrm{Gr}_2(V) \times \mathrm{Gr}_3(V) \mid l_1 \subset L,\ l_1 \subset l_2 \subset l_3,\ L \subset l_3 \}. \tag{7}$$

The space F_L is a smooth manifold of dimension 3.

We will analyze the two projections

$$\pi_2 : F_L \to \mathrm{Gr}_2(V) : (l_1, l_2, l_3) \mapsto l_2, \tag{8}$$

$$\pi_{1,3} : F_L \to \mathrm{Gr}_1(L) \times \mathrm{Gr}_1(V/L) : (l_1, l_2, l_3) \mapsto (l_1, l_3/L). \tag{9}$$

The projection $\pi_{1,3} : F_L \to \mathrm{Gr}_1(L) \times \mathrm{Gr}_1(V/L)$ is surjective. The fiber above a point $(l_1, l_3/L)$ is diffeomorphic to $\mathrm{Gr}_1(l_3/l_1)$. This shows $\pi_{1,3}$ is a $\mathbb{P}^1$ bundle over $\mathrm{Gr}_1(L) \times \mathrm{Gr}_1(V/L)$.

For every point $(l_1, l_2, l_3) \in F_L$ the intersection of l_2 and L is non-empty. This implies that the image of π_2 is contained in Σ_L and it is not difficult to see that $\pi_2 : F_L \to \Sigma_L$ is surjective. At the points $l_2 \neq L$ in the image of π_2 we have $\pi_2^{-1}(l_2) = \{ (l_2 \cap L, l_2, l_2 + L) \}$. So π_2 is injective over the complement of L in $\mathrm{Gr}_2(V)$. The rank of $T\pi_2$ over this complement is 3*. For the special point L we have*

$$\pi_2^{-1}(L) = \{ (l_1, L, l_3) \in F_L \mid l_1 \in \mathrm{Gr}_1(L),\ L \subset l_3 \in \mathrm{Gr}_3(V) \}.$$

This shows that $T\pi_2$ has rank 1 at the points in F_L that project to L. The map

$$\pi_2^{-1}(L) \to \mathrm{Gr}_1(L) \times \mathrm{Gr}_1(V/L) : (l_1, L, l_3) \mapsto (l_1, l_3/L)$$

is an isomorphism. This shows that the inverse image $\pi_2^{-1}(L)$ defines a special section of the bundle $\pi_{1,3} : F_L \mapsto \mathrm{Gr}_1(L) \times \mathrm{Gr}_1(V/L)$.

Let $F'_L = \{ (l_1, l_2, l_3) \in F_L \mid l_2 \neq L \}$ and let π'_2 and $\pi'_{1,3}$ be the restrictions of π_2 and $\pi_{1,3}$, respectively, to the bundle F'_L. The fiber of $\pi'_{1,3}$ above a point $(l_1, l_3/L)$ is isomorphic to $\mathrm{Gr}_1(l_3/l_1) \setminus (L/l_1) \cong \mathbb{P}^1 \setminus \{ 0 \}$. This gives $\pi'_{1,3} : F'_L \to \mathrm{Gr}_1(L) \times \mathrm{Gr}_1(V/L)$ the structure of an affine line bundle.

For any (local) section σ of the bundle $\pi'_{1,3}$ we can consider the composition $\pi'_2 \circ \sigma : \mathrm{Gr}_1(L) \times \mathrm{Gr}_1(V/L) \to \mathrm{Gr}_2(V)$. The map is embedding since $\pi'_2 : F'_L \to \mathrm{Gr}_2(V)$ has rank 3 and is injective. Global sections of this bundle exist. Take for example a transversal 2*-plane M such that $V = L \oplus M$. A global section of $\pi'_{1,3}$ is given by $(l_1, l_3) \mapsto (l_1, l_1 + M \cap l_3, l_3)$. The hyperbolic surface defined* *by the composition of this section with π_2' is the standard hyperbolic torus* $\mathrm{Gr}_2(V, K)$ *for the product structure* K *defined by* $V = L \oplus M$*. After a choice of global section the line bundle* F_L' *becomes a rank one vector bundle over* $\mathrm{Gr}_1(L) \times \mathrm{Gr}_1(V/L)$*. The sections of this bundle can locally be parameterized by exactly one function of two variables. The global sections define compact geometrically flat surfaces of type (2′, 3′).*

Type (1, 1). We will prove that any connected geometrically flat surface S in $\mathrm{Gr}_2(V)$ of type (1, 1) is locally given by the hyperbolic lines for a unique product structure on V. Note that a surface of type (1, 1) is both a surface of type (2′, 3′) and of type (3′, 2′). For a surface of type (2′, 3′) there is a unique 2-plane L_+ that satisfies (6). Since the surface is of type (3′, 2′) as well, there is also a unique 2-plane L_- with a relation similar to (6). It is not difficult to prove that S must be equal to a subset of the standard hyperbolic torus $\mathrm{Gr}_2(V, K)$ defined by the product structure K that defines the decomposition $V = L_+ \oplus L_-$.

4.5 Normal form calculations

In this section we calculate a normal form for the hyperbolic surfaces. The group acting on the surface is the group of conformal isometries of $\mathrm{Gr}_2(V)$. With the action of this group we will bring the Taylor expansion of a parameterization of the surface into normal form. For generic surfaces the normal form construction leads to a complete description of the invariants of the surface. There is also a geometric interpretation of this normal form calculation in terms of moving frames. This geometric picture is presented in [9, §2.3.4] (hyperbolic surfaces) and [1, §4.3, 6.7] (elliptic surfaces) and can be used to make a connection to the local invariants of partial differential equations.

Zero and first order. We want to bring a hyperbolic surface S in $\mathrm{Gr}_2(V)$ into a normal form using the group $\mathrm{GL}(V)$. We could also use the projective group $\mathbb{P}\,\mathrm{GL}(V)$ since the scalar multiplications do not act on $\mathrm{Gr}_2(V)$. Since the group acts transitively on the points in $\mathrm{Gr}_2(V)$ and on the hyperbolic tangent spaces at that point, we can always choose a basis e_1, e_2, e_3, e_4 for V such that the point $L \in S$ is given by $\mathbb{R}e_1 + \mathbb{R}e_2$ and the tangent space to the surface at L is given by the linear maps in $\mathrm{Lin}(L, V/L)$ that are diagonal matrices with respect to the bases e_1, e_2 for L and $e_3 + L, e_4 + L$ for V/L.

In the local coordinates introduced in Section 2.3 the surface can be parameterized as

$$(p, q) \mapsto A = \begin{pmatrix} p & q(p, s) \\ r(p, s) & s \end{pmatrix}. \tag{10}$$

The special point L corresponds to the zero matrix. We will bring the surface in normal form by constructing a normal form for the Taylor expansions of $q(p,s)$ and $r(p,s)$. The normalization at order zero was the choice of special point L. This normalization corresponds to $q(0,0)=0$ and $r(0,0)=0$. The normalization at order one was the choice of tangent space to S at L. This corresponds to $q=\mathcal{O}(p,s)^2$, $r=\mathcal{O}(p,s)^2$.

The group $\mathrm{GL}(V)$ can be parameterized by the 4×4 matrices $\begin{pmatrix}\tilde\alpha & \tilde\beta \\ \tilde\gamma & \tilde\delta\end{pmatrix}$, with $\tilde\alpha,\tilde\beta,\tilde\gamma,\tilde\delta$ all 2×2-matrices. The subgroup H_0 that leaves invariant L is given by the matrices with $\tilde\gamma=0$. We compute the action of H_0 on the tangent space $T_L S$. Let

$$g=\begin{pmatrix}\tilde\alpha & \tilde\beta \\ 0 & \tilde\delta\end{pmatrix}.$$

Then g acts on A as $g\cdot A=\tilde\delta A(\tilde\alpha+\tilde\beta A)^{-1}$. On the tangent space to the Grassmannian this induces the action $X\mapsto \tilde\delta X\tilde\alpha^{-1}$. This conformal action is transitive on the hyperbolic planes and we can always arrange that the tangent space to the surface at L consists of diagonal matrices.

The structure group that leaves invariant L and $T_L S$ is the group H_1 of matrices

$$\begin{pmatrix}\alpha & \tilde\beta \\ 0 & \delta\end{pmatrix}\in \mathrm{GL}(V), \tag{11}$$

with either α,δ both diagonal or α,δ both anti-diagonal. This group has dimension 8 (or dimension 7 if we are working with the projective group) and 2 connected components.

Second order. The space of second order contacts to a hyperbolic surface for which the first order part is in normal form, has dimension 6. The action of the group H_1 induces an action on this space by affine transformations. If we use the local coordinates (10), then the first order normalizations correspond to

$$\begin{aligned} q &= q_{11}p^2/2+q_{12}ps+q_{22}s^2/2+\mathcal{O}(p,s)^3,\\ r &= r_{11}p^2/2+r_{12}ps+r_{22}s^2/2+\mathcal{O}(p,s)^3. \end{aligned}$$

The action of H_1 on A is given by

$$A\mapsto \delta A(\alpha+\tilde\beta A)^{-1}=\delta A\alpha^{-1}-\delta A\alpha^{-1}\tilde\beta A\alpha^{-1}+\mathcal{O}(p,s)^3.$$

We will calculate the action of the connected component of the group H_1. The action of the other component can be calculated in a similar fashion. We write

$$\alpha = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}, \quad \tilde{\beta} = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}, \quad \delta = \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix}.$$

Working out this action using p, s as coordinates and only keeping terms of order 2 and lower yields

$$A \mapsto \tilde{A} = \begin{pmatrix} \delta_1 \alpha_1^{-1} p - \delta_1 \alpha_1^{-2} p^2 \beta_{11} & \delta_1 \alpha_2^{-1} q - \delta_1 \alpha_2^{-1} \alpha_1^{-1} \beta_{12} ps \\ \delta_2 \alpha_1^{-1} r - \delta_2 \alpha_1^{-1} \alpha_2^{-1} \beta_{21} ps & \delta_2 \alpha_2^{-1} s - \delta_2 \alpha_2^{-2} \beta_{22} s^2 \end{pmatrix} + \mathcal{O}(p, s)^3.$$

We use $\tilde{p} = \delta_1 \alpha_1^{-1} p - \delta_1 \alpha_1^{-2} \beta_{11} p^2$ and $\tilde{s} = \delta_2 \alpha_2^{-1} s - \delta_2 \alpha_2^{-2} \beta_{22} s^2$ as new local coordinates. Since $\tilde{p}, \tilde{s}$ are diagonal in p, s up to first order, this preserves the normal form. We can express $\tilde{q}$ and $\tilde{r}$ in the new coordinates $\tilde{p}$, $\tilde{s}$; the final result is

$$\begin{aligned} \tilde{q}_{11} &= (\alpha_1)^2 \alpha_2^{-1} \delta_1^{-1} q_{11}, & \tilde{q}_{12} &= \alpha_1 \delta_2^{-1} q_{12} - \delta_2^{-1} \beta_{12}, & \tilde{q}_{22} &= \delta_1 \delta_2^{-2} \alpha_2 q_{22}, \\ \tilde{r}_{11} &= \delta_2 \delta_1^{-2} \alpha_1 r_{11}, & \tilde{r}_{12} &= \alpha_2 \delta_1^{-1} r_{12} - \delta_1^{-1} \beta_{21}, & \tilde{r}_{22} &= \alpha_1^{-1} (\alpha_2)^2 \delta_2^{-1} r_{22}. \end{aligned} \tag{12}$$

The action is indeed by affine transformations. Note that the group coefficients β_{11}, β_{22} do not appear in these expressions, so this part of the group does not act on the second order contact. Using the group parameters β_{12}, β_{21} we can always arrange that $\tilde{q}_{12} = \tilde{r}_{12} = 0$. This normalization reduces the identity component of the structure group to the group of matrices

$$g = \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix},$$

with $\alpha, \delta \in D^*$ and $\beta \in D$. Here D is the algebra of diagonal 2×2-matrices.

In [1, Section 4.4] a normal form calculation is done for elliptic surfaces in the Grassmannian. McKay finds similar normalizations, but formulates the structure groups in terms of complex numbers. In [9] the hyperbolic surfaces are analyzed with an equivalent to the complex number, the algebra of *split-complex numbers* [12]. The algebra of split-complex numbers are isomorphic to the algebra of diagonal 2×2-matrices.

On the remaining four coefficients the generic orbits have dimension three. There is one invariant given by

$$I = \frac{q_{11} r_{22}}{r_{11} q_{22}}. \tag{13}$$

The invariant is a rational function in the coefficients of the second order jets of a hyperbolic surface. If $r_{11}q_{22} = 0$ but $q_{11}r_{22} \neq 0$, then we say the invariant takes the value ∞. If both $q_{11}r_{22} = 0$ and $r_{11}q_{22} = 0$, then this invariant is not well-defined (by making small perturbations the invariant can have any possible value).

Remark 19 *We will analyze the action of the other component of H_1 on the second order coefficients. Let*

$$g = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in H_1.$$

The action on a surface in local coordinates is

$$A = \begin{pmatrix} p & q(p,s) \\ r(p,s) & s \end{pmatrix} \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p & q(p,s) \\ r(p,s) & s \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} s & r(p,s) \\ q(p,s) & p \end{pmatrix}.$$

If we write $\tilde{p} = s$, $\tilde{s} = p$, $\tilde{q} = r$, $\tilde{r} = q$ and assume that q, r are normalized and of the form $q = q_{11}p^2/2 + q_{12}ps + q_{22}s^2/2$, $r = r_{11}p^2/2 + r_{12}ps + q_{22}r^2/2$, then

$$\tilde{q}_{11} = r_{22}, \quad \tilde{q}_{12} = r_{12}, \quad \tilde{q}_{22} = r_{11},$$
$$\tilde{r}_{11} = q_{22}, \quad \tilde{r}_{12} = q_{12}, \quad \tilde{r}_{22} = q_{11}.$$

The invariant I is unchanged by this transformation, i.e.,

$$I = \frac{\tilde{q}_{11}\tilde{r}_{22}}{\tilde{r}_{11}\tilde{q}_{22}} = \frac{q_{11}r_{22}}{r_{11}q_{22}}.$$

So I is really invariant under the full group H_1.

Third and higher order. We will conclude the normal form calculations by showing that for generic structures (all terms $q_{11}, r_{22}, r_{11}, q_{22}$ unequal to zero, or equivalently the invariant I is well-defined, non-zero and finite) the projective group acts effectively. If we are at a generic point, then we can normalize the second order coefficients to $q_{12} = r_{12} = 0$, $q_{11} = r_{11} = q_{22} = 1$ and $r_{22} = I$. The structure group reduces to the group H_3 consisting of matrices

$$g = \phi \begin{pmatrix} I & \beta \\ 0 & I \end{pmatrix} \in \mathrm{GL}(V)$$

with $\phi \in \mathbb{R}^*$ and $\beta = \mathrm{diag}(\beta_{11}, \beta_{22}) \in D$. The scalar factor ϕ is not important since the scalar multiples of the identity are in the kernel of the action.

The action on the third order part is relatively easy to calculate because the structure group has reduced to such a small group. The action of g on the matrix A is

$$g : A \mapsto \tilde{A} = A(I + bA)^{-1} = A - A\beta A + A\beta A\beta A + \mathcal{O}(|A|)^4. \qquad (14)$$

At the special point L we have $A = 0$ and the first order part of A is diagonal. Therefore we can write $A = A_1 + A_2 + A_3 + \mathcal{O}(p,s)^4$ with $A_1 = \begin{pmatrix} p & 0 \\ 0 & s \end{pmatrix} \in D$ and the second and third order parts A_2 and A_3 anti-diagonal and homogeneous of degree 2 and 3 in the parameters p, s, respectively. In a similar way we can expand $\tilde{A}$ into $\tilde{A}_1 + \tilde{A}_2 + \tilde{A}_3 + \mathcal{O}(\tilde{p},\tilde{s})^4$, with $\tilde{A}_1 = \begin{pmatrix} \tilde{p} & 0 \\ 0 & \tilde{s} \end{pmatrix}$.

After some calculations we find that $\tilde{p} = p - \beta_{11}p^2 + \mathcal{O}(p)^2$ and $\tilde{s} = s - \beta_{22}s^2 + \mathcal{O}(s)^2$. The coefficients in the second order part are unchanged, so

$$\tilde{A}_2 = \begin{pmatrix} 0 & \tilde{q}_2 \\ \tilde{r}_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & q_{11}\tilde{p}^2/2 + q_{22}\tilde{s}^2/2 \\ r_{11}\tilde{p}^2/2 + r_{22}\tilde{s}^2/2 & 0 \end{pmatrix}. \qquad (15)$$

Calculating the third order part we find that

$$\begin{aligned}\tilde{A}_3 = &\begin{pmatrix} 0 & \begin{matrix} q_{111}\tilde{p}^3/3 + q_{112}\tilde{p}^2\tilde{s} \\ \quad + q_{122}\tilde{p}\tilde{s}^2 + q_{222}\tilde{s}^3/3 \end{matrix} \\ \begin{matrix} r_{111}\tilde{p}^3/3 + r_{112}\tilde{p}^2\tilde{s} \\ \quad + r_{122}\tilde{p}\tilde{s}^2 + r_{222}\tilde{s}^3/3 \end{matrix} & 0 \end{pmatrix} \\ &+ \begin{pmatrix} 0 & q_{11}\beta_{11}\tilde{p}^3 + q_{22}\beta_{22}\tilde{s}^3 \\ r_{11}\beta_{11}\tilde{p}^3 + r_{22}\beta_{22}\tilde{s}^3 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \tilde{p}\beta_{11}\tilde{q}_2 + \tilde{q}_2\beta_{22}\tilde{s} \\ \tilde{s}\beta_{22}\tilde{r}_2 + \tilde{r}_2\beta_{11}\tilde{p} & 0 \end{pmatrix}.\end{aligned}$$

For the action of the projective group on the third order coefficients to be effective, it is necessary and sufficient that at least one out of the four coefficients q_{11}, r_{11}, q_{22} and r_{22} is non-zero. For a generic point this action is effective. Hence by normalizing two suitable third order coefficients, the structure group reduces to the scalar multiplications. The remaining six third order coefficients are invariants for the surface.

For higher order contact at each order n there are precisely $2(n+1)$ more derivatives of q and r. Since the structure group already reduced to the scalar multiplications at order 3 (for generic structures), we find at each order precisely $2(n+1)$ additional invariants.

Example 20 (Invariants for geometrically flat surfaces)

- *In local coordinates for the Grassmannian define a geometrically flat surface of type (2′, 2′) by the matrices*

$$\begin{pmatrix} p & q(s) \\ r(p) & s \end{pmatrix},$$

with $r(p) = r_{11}p^2/2 + \mathcal{O}(p^3)$, $q(s) = q_{22}s^2/2 + \mathcal{O}(s^3)$. *The points on the characteristic line* $p = p_0$ *all have the line* $l_1(p_0) = \mathbb{R}(1, 0, p_0, r(p_0))^{\mathrm{T}}$ *in common. The points on the characteristic line* $s = s_0$ *all have the line* $l_1(s_0) = \mathbb{R}(0, 1, q(s_0), s_0)^{\mathrm{T}}$ *in common. If the surface is generic enough, i.e.,* $r_{11}q_{22} \neq 0$, *the invariant* I *is well-defined and equal to zero.*
- *In local coordinates define the surface* S *by the matrices of the form*

$$\begin{pmatrix} a & 0 \\ \phi(a, b) & b \end{pmatrix}.$$

This is a geometrically flat surface of type (2′, 3′). All points on the characteristic line $b =$ *constant have the line* $l_1 = (0, 1, 0, 1)^{\mathrm{T}}$ *in common. The points in the lines* $a =$ *constant are all contained in the 3-dimensional subspace spanned by the vectors* $(1, 0, a, 0)^{\mathrm{T}}, (0, 1, 0, 0)^{\mathrm{T}}, (0, 0, 0, 1)^{\mathrm{T}}$. *The special point* L *defined in equation* (6) *is equal to* $\mathbb{R}(0, 1, 0, 0)^{\mathrm{T}} + \mathbb{R}(0, 0, 0, 1)^{\mathrm{T}}$. *The coefficients* r_{11}, r_{12} *and* r_{22} *are zero and hence* I *is not well-defined since both the numerator and the denominator are zero.*
- *In local coordinates define the geometrically flat surface of type (3′, 3′) by*

$$\begin{pmatrix} p & q(p) \\ r(s) & s \end{pmatrix},$$

with $r(s) = r_{22}s^2/2 + \mathcal{O}(s)^3$, $q(p) = q_{11}p^2/2 + \mathcal{O}(p^3)$. *If the surface is generic enough, i.e.,* $r_{22}q_{11} \neq 0$, *the invariant* I *is well-defined and has value* ∞.

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