The Borel-Weil Theorem

Pieter T. Eendebak

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Contents

Introduction

This is my small thesis as part of the Mathematics program at the Utrecht University.

The most important subject of this thesis is the Borel-Weil theorem. This theorem gives a construction of a representation of a compact Lie group G with highest weight λ using only this highest weight. Because every irreducible representation is uniquely characterized by its highest weight and all weights can be determined by looking at the Lie algebra of G, this construction yields all irreducible representations of G. A short sketch of the construction that is used, is given in the first section of chapter 3.

The thesis consists out of three chapters. The first chapter deals with holomorphic sections of a vector bundle. We prove that the space of holomorphic sections, of a vector bundle over a compact complex manifold, is finite-dimensional. This is needed in the proof of the Borel-Weil theorem, where we use it to prove that the representation space of an induced representation is finite-dimensional. The second chapter is entirely about induced representations. Induced representations are a mathematical construction with various applications (such as in quantum mechanics, see [6]). We will only use induced representations in the last chapter to prove the Borel-Weil theorem for the group SL(n,$\mathbb{C}$). For this proof we also need some other concepts (such complexification of groups, weights and characters of a torus, Bruhat-decomposition). From these topics only some results are given and most proofs are omitted.

To be able to understand the full text, the reader should know the basics of analysis, Lie groups and Lie algebras (especially the root-space decomposition of a Lie algebra) and complex function theory in one variable.

Chapter 1

Holomorphic vector bundles

The main goal of this chapter is to prove theorem 1.3.3, which states that the space of holomorphic sections of a holomorphic vector bundle over a compact complex manifold is finite-dimensional. For trivial vector bundles over a complex manifold the space of holomorphic sections will prove to have a very simple structure. The theorem will be needed later on in the proof of the Borel-Weil theorem.

1.1 Vector bundles

To define holomorphic vector bundles, we need complex manifolds and holomorphic (or complex-analytic) functions on complex manifolds. The basic definitions of these are given in appendix A.1.

Definition 1.1.1. A (holomorphic or complex-analytic) vector bundle (E, π) of rank r over a manifold M is a (complex-analytic) mapping $\pi : E \to M$ of (complex) manifolds for which every fiber $E_m = \pi^{-1}(m)$ has the structure of a (complex) vector space of dimension r. Moreover we can cover M with open sets U such that there is a mapping $\rho : \pi^{-1}(U) \to V$ for which

- $\tilde{\rho} : \pi^{-1}(U) \to U \times V : x \to (\pi(x), \rho(x))$ is a (complex-analytic) diffeomorphism.
- For every $m \in M$ the mapping $\rho : E_m \to V$ is a linear isomorphism.

The pair (U, ρ) is called a local trivialisation of the vector bundle.

A section s of a vector bundle (E, π) over M is a function $s : M \to E$ for which $\pi \circ s = \mathrm{id}_M$. The sections of a vector bundle are usually a very large class of functions. Therefore in this section we only look at the holomorphic sections of a complex vector bundle E which we denote by $\mathcal{O}(E)$ or $\mathcal{O}(M, E)$. This class will prove to be much smaller.

Definition 1.1.2. Let G be a Lie group, H a closed subgroup and $(\mathcal{W}, \pi)$ a vector bundle over G/H. $\mathcal{W}$ is called a *homogeneous* vector bundle over G/H if G has a smooth action $\cdot$ on $\mathcal{W}$ such that, for every $g \in G$ the following diagram is commutative.

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{g\cdot} & \mathcal{W} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ G/H & \xrightarrow{l_g} & G/H \end{array}$$

So the action of $g \in G$ on $\mathcal{W}$ sends the fiber $\pi^{-1}(xH)$ to the fiber $\pi^{-1}(gxH)$.

1.2 Complex function theory

Because we do not want to restrict ourselves to one-dimensional manifolds, we will first consider holomorphic functions of several complex variables. We give the most important definitions and theorems of complex function theory in more than one variable. Almost all of these definitions and theorems are straightforward generalizations of similar definitions and theorems in the one variable case.

A multi-index ν is an element of $\mathcal{F} = (Z_{\geq 0})^n$. For a point $\mathbf{z} = (z_1, z_2, \ldots, z_n)$ in $\mathbb{C}^n$ and a multi-index $\nu = (\nu_1, \nu_2, \ldots, \nu_n) \in \mathcal{F}$ we write $\mathbf{z}^\nu = z_1^{\nu_1} z_2^{\nu_2} \cdots z_n^{\nu_n}$. In this notation we can write a complex polynomial as

$$p(\mathbf{z}) = \sum_{\nu_1=0,\ldots,\nu_n=0}^{m_1,\ldots,m_n} a_{\nu_1,\ldots,\nu_n} z_1^{\nu_1} \cdots z_n^{\nu_n} = \sum_{\nu=0}^{m} a_\nu \mathbf{z}^\nu.$$

In this way we can easily write down complex polynomials and power series.

A formal power series (around $\mathbf{z}'$) is an expression of the form

$$\sum_{\nu \in \mathcal{F}} a_\nu (\mathbf{z} - \mathbf{z}')^\nu \tag{1.1}$$

where a_ν is a complex number for every multi-index ν and $\mathbf{z}' \in \mathbb{C}^n$ is fixed. This power series is a generalization of the power series in one variable. To give a precise meaning of this series as a complex number (for a fixed value of $\mathbf{z}$), we must specify in which order we must do the (infinite) summation.

Definition 1.2.1. Let $\mathbf{z} \in \mathbb{C}^n$. We say that the power series (1.1) converges at the point $\mathbf{z}$ to $c \in \mathbb{C}$ if $\forall \epsilon > 0$ there is a finite set $I_0 \subset \mathcal{F}$ such that for all sets I with $I_0 \subset I \subset \mathcal{F}$

$$\left| \sum_{\nu \in I} a_\nu (\mathbf{z} - \mathbf{z}')^\nu - c \right| < \epsilon.$$

We say that a power series uniformly converges to a function f on $M \subset \mathbb{C}^n$ if $\forall \epsilon > 0$ there is a finite $I_0 \subset \mathcal{F}$ such that for all sets I with $I_0 \subset I \subset \mathcal{F}$

$$\left| \sum_{\nu \in I} a_\nu (\mathbf{z} - \mathbf{z}')^\nu - f(\mathbf{z}) \right| < \epsilon$$

for all $\mathbf{z} \in M$.

Definition 1.2.2. Let $B \subset \mathbb{C}^n$ be a region (an open subset of $\mathbb{C}^n$) and $f : \mathbb{C}^n \to \mathbb{C}$ a complex function. The function f is called holomorphic if for every point $\mathbf{z}_0$ in B there is a power series

$$\sum_{\nu \in \mathcal{F}} a_\nu (\mathbf{z} - \mathbf{z}_0)^\nu$$

which converges to f on a neighbourhood of $\mathbf{z}_0$.

This definition is consistent with other definitions of holomorphic functions (such as a definition using the Cauchy-Riemann equations, see [3]).

In one-variable complex function theory one of the main theorems is the Cauchy integral formula. We will need the generalization of this theorem to several variables. We wil give the main results without giving a proof. The proofs can be found in [3] or other books on multi-variable complex analysis.

Definition 1.2.3. Let $\mathbf{r} = (r_1, \ldots, r_n) \in \mathbb{R}^n_{>0}$ and $\mathbf{z}' \in \mathbb{C}^n$. The polydisc D about $\mathbf{z}'$ with radius $\mathbf{r}$ is defined as

$$D_{\mathbf{r}}(\mathbf{z}') = \{\, \mathbf{z} \in \mathbb{C}^n \mid |z_k - z'_k| < r_k \text{ for } 1 \leq k \leq n \,\}.$$

The distinguished boundary of D is defined by

$$\partial_0 D = \{\, \mathbf{z} \in C^n \mid |z_k - z_k'| = r_k \text{ for } 1 \leq k \leq n \,\}.$$

Note that the distinguished boundary $\partial_0 D$ is in general not equal to the ordinary boundary $\partial D = \bar{D}$ of D.

The Cauchy integral formula of one-variable theory states that for a holomorphic function f and a simple closed curve γ we have

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - z}\, d\xi$$

for all z inside the curve γ. The generalization of this is an integral over the boundary of a polydisc.

Theorem 1.2.1 (The Cauchy Integral Formula)**.** *Let B be a region in $\mathbb{C}^n$ and $D = D_{\mathbf{r}}(\xi)$ a polydisc with $\partial D \subset B$. Let $f : B \to \mathbb{C}$ be a holomorphic function.*

Then for $\mathbf{z} \in D$

$$\begin{aligned} f(\mathbf{z}) &= (\frac{1}{2\pi i})^n \int_{\partial_0 D} \frac{f(\mathbf{x})}{(x_1 - z_1)\cdots(x_n - z_n)} d\mathbf{x} \\ &= (\frac{1}{2\pi i})^n \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{f(\xi_1 + r_1 e^{i\theta_1}, \ldots, \xi_n + r_n e^{i\theta_n})}{(\xi_1 + r_1 e^{i\theta_1} - z_1)\cdots(\xi_n + r_n e^{i\theta_n} - z_n)}\, d\theta_1 d\theta_2 \ldots d\theta_n. \end{aligned}$$

Theorem 1.2.2. *Let B be a region in $\mathbb{C}^n$ and $(f_j)_{j=1}^\infty$ a sequence of holomorphic functions on B. If the f_j convergence uniformly to a function f on B, then f is holomorphic on B.*

Theorem 1.2.3. *Let $f : \mathbb{C}^m \to \mathbb{C}$ be a holomorphic function and suppose $f(\lambda\mathbf{z}) = \lambda^n f(\mathbf{z})$ for all $\lambda \in \mathbb{C}, \mathbf{z} \in \mathbb{C}^m$.*

Then f is zero if $n < 0$ or a homogeneous polonomial of degree n if $n \geq 0$.

Proof. If $n < 0$ we see that f must be zero everywhere, otherwise f is not well defined in 0.

Suppose $n \geq 0$. Let ν be a multi-index with $|\nu| \leq n$, then $\partial^\nu f = \frac{\partial^{|\nu|} f}{\partial z_1^{\nu_1} \ldots \partial z_n^{\nu_n}}$ satisfies $(\partial^\nu f)(\lambda\mathbf{z}) = \lambda^{n-|\nu|}\partial^\nu f(\mathbf{z})$. We prove this by induction on $|\nu|$.

- For $|\nu| = 0$ the statement $f(\lambda\mathbf{z}) = \lambda^n f(\mathbf{z})$ is true by assumption on f.
- Suppose $(\partial^\nu f)(\lambda\mathbf{z}) = \lambda^{n-|\nu|}\partial^\nu f(\mathbf{z})$ for all $|\nu| < d$ and we have a multi-index μ with $|\mu| = d > 0$. We may assume $\mu_1 > 0$ (or arrange this by permuting indices). Take for ν the multi-index with $\nu_1 = \mu_1 - 1$ and $\nu_j = \mu_j$ for $1 < j \leq m$. Now by the induction hypothesis we have

  $$(\partial^\nu f)(\lambda\mathbf{z}) = \lambda^{n-|\nu|}(\partial^\nu f)(\mathbf{z})$$

  and thus

  $$\begin{aligned} (\partial^\nu f)(\lambda\mathbf{z}) &= \lambda^{n-|\nu|}\partial^\nu f(\mathbf{z}) \\ \frac{\partial}{\partial z_1}((\partial^\nu f)(\lambda\mathbf{z})) &= \frac{\partial}{\partial z_1}\lambda^{n-|\nu|}\partial^\nu f(\mathbf{z}) \\ (\frac{\partial}{\partial z_1}\partial^\nu f)(\lambda\mathbf{z})\lambda &= \lambda^{n-|\nu|}\frac{\partial}{\partial z_1}\partial^\nu f(\mathbf{z}) \\ (\partial^\mu f)(\lambda\mathbf{z}) &= \lambda^{n-|\nu|-1}\partial^\mu f(\mathbf{z}). \end{aligned}$$

  This means that $(\partial^\mu f)(\lambda\mathbf{z}) = \lambda^{n-|\mu|}(\partial^\mu f)(\mathbf{z})$.

So now we have for every ν with $|\nu| = n$ that $(\partial^\nu f)(\lambda \mathbf{z}) = \lambda^{n-|\nu|}(\partial^\nu f)(\mathbf{z}) = (\partial^\nu f)(\mathbf{z})$. So $(\partial^\nu f)(\mathbf{z}) = (\partial^\nu f)(0)$, hence $\partial^\nu f$ is a constant function. From the power series of f we see that $f = \sum_{|\nu|\leq n} a_\nu \mathbf{z}^\nu$, so f is a polynomial of degree $\leq n$. It now follows from the transformation rule that $a_\nu = 0$ if $|\nu| \neq n$. □

The last theorem and its corollary give some estimates on the (partial) derivatives of a holomorphic function.

Theorem 1.2.4 (Cauchy's estimates)**.** *Let f be a holomorphic function on the polydisc $D = D_{\mathbf{r}}(\mathbf{z})$. Then for every multi-index α we have*

$$|\partial^\alpha f(\mathbf{z})| \leq |\alpha|! \frac{M}{\mathbf{r}^\alpha}$$

where $M = \sup_{\mathbf{z}\in D} f(\mathbf{z})$.

Proof. The proof of the theorem can be found in [3]. The proof follows from the Cauchy integral formula by differentiation under the integral sign and some estimates on the maximum of f. □

Corollary 1.2.5. *Let f be a holomorphic function on the polydisc $D = D(\mathbf{z}, r)$. Then for every $1 \leq k \leq n$*

$$\left| \frac{\partial f}{\partial z_k}(\mathbf{z}) \right| \leq \frac{M}{r}$$

where $M = \sup_{\mathbf{z}\in D} f(\mathbf{z})$.

1.3 Holomorphic vector bundles

We start with some general definitions and theorems which we will need in this section. For two points x, y in $\mathbb{R}^n$ or $\mathbb{C}^n$ we write $d(x, y) = ||x-y||$, so $d(x, y)$ is just the Euclidian distance between x and y. For two sets U and V we define $d(U, V) = \inf_{x\in U, y\in V} d(x, y)$.

Definition 1.3.1. Suppose M is a set of functions from $\mathbb{C}^n$ to $\mathbb{C}$. The set M is called equicontinuous on $U \subset \mathbb{C}^n$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that for all $x, y \in U$

$$||x - y|| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$$

for all $f \in M$.

We can also use this definition of equicontinuity for functions $f : X \to Y$ where X and Y are arbitrary metric spaces. For an equicontinuous set of functions M we have the following theorem:

Theorem 1.3.1 (Ascoli's theorem)**.** *Let X be a compact metric space and let M be a subset of the space $C(X, \mathbb{C}^n)$ (the space of continuous functions $X \to \mathbb{C}^n$ with the supremum norm). Then M is compact if and only if M is closed, bounded and equicontinuous.*

Proof. The proof can be found in various forms in many books, see for example [7]. □

To prove the main theorem of this section, we need the following lemma.

Lemma 1.3.2. *Let U be a region of $\mathbb{C}^n$, $x \in U$ and $V = \bar{B}(x, r_V)$ a closed ball for which $d(\mathbb{C}^n \setminus U, V) \geq \eta > 0$. Let $(f_j)_{j=1}^\infty$ be a bounded sequence of holomorphic functions on U. Then $(f_j|_V)_{j=1}^\infty$ is an equicontinuous sequence of functions on V.*

Proof. Take $M = \sup_{1\le j\le\infty, z\in U} f_j(z)$. From corollary 1.2.5 we see that $\frac{\partial f_j}{\partial z_k}(\mathbf{z}) \le M/\eta$ for all j and $1 \le k \le n$. We write $\frac{\partial f_j}{\partial z}$ for derivative of f_j with respect to z, so $\frac{\partial f_j}{\partial z} = (\frac{\partial f_j}{\partial z_1}, \ldots \frac{\partial f_j}{\partial z_n})$.

Let $\epsilon > 0$ be given. Choose $\delta = \frac{\eta}{M}\epsilon$. Suppose $x, y \in V$ and $|x - y| < \delta$. Let γ be the straight line between x and y. Then for every $j \ge 1$:

$$\begin{aligned} |f_j(x) - f_j(y)| &\le |\int_\gamma \frac{\partial f_j}{\partial z} \cdot \gamma'(t)\, dt| \le \int_\gamma |\frac{\partial f_j}{\partial z} \cdot \gamma'(t)|\, dt \\ &\le \int_\gamma \frac{M}{\eta} |\gamma'(t)|\, dt = \frac{M}{\eta}|x - y| < \epsilon. \end{aligned}$$

This means that the family $(f_j)_{j=1}^\infty$ is equicontinuous. □

Theorem 1.3.3. *Let M be a compact, complex manifold and let (E, π) be a holomorphic vector bundle over M. Then the space of holomorphic sections $\mathcal{O}(M, E)$ is finite-dimensional.*

Proof. First we define $m = \dim M$ and $n = \operatorname{rank} E$.

- First we will introduce a norm on the space of continuous sections. The restriction of this norm to $\mathcal{O}(M, E)$ will be a norm for which $\mathcal{O}(M, E)$ is a Banach space.

  For each fiber $E_x = \pi^{-1}(x)$ we will choose a norm $|| \cdot ||_x$ such that this norm depends continuously on x. For each point $x \in M$, choose a neighbourhood U_x of x such that there is a chart (U_x, κ_x) to an open region $\tilde{U}_x \subset \mathbb{C}^m$ and a trivialisation (U_x, ρ_x) of the vector bundle. The mapping $\tilde{\rho}_x : v \to ((\kappa_x \circ \pi)(v), \rho_x(v))$ is a diffeomorphism from $\pi^{-1}(U_x)$ to $\tilde{U}_x \times \mathbb{C}^n$.

  $$\begin{array}{ccc} \pi^{-1}(U_x) & \xrightarrow{\pi\times\rho_x} & U_x \times \mathbb{C}^n \\ \downarrow{\scriptstyle \pi} & & \\ U_x & \xrightarrow[\kappa_x]{} & \tilde{U}_x \subset \mathbb{C}^m \end{array}$$

  Because M is compact we can choose from the covering $(U_x)_{x\in M}$ a finite subcovering $(U_\alpha)_{\alpha\in I}$ where I is a certain index set. Now choose a partition of unity ϕ_α subordinate to (U_α). So $(\phi_\alpha)_{\alpha\in J}$ is a collection of compactly supported smooth functions $M \to \mathbb{C}$ such that $\operatorname{supp} \phi_\alpha \subset U_\alpha$ and $\sum_{\alpha\in J} \phi_\alpha = 1$ on M.

  Now define $||v||_x = \sum_\alpha \phi_\alpha(x)|\rho_\alpha(v)|$. This defines a norm on V_x for each $x \in M$. For a continuous section s of E we can now take

  $$||s|| = \sup_{x\in M} ||s(x)||_x.$$

  Because M is compact and $|| \cdot ||_x$ is continuous this is well defined. It is easy to check that this defines a norm on $C(M, V)$. We can use this norm to define a metric on the space $C(M, V)$ by defining the distance between two functions f and g as $d(f, g) = ||f - g||$. So the space $C(M, V)$ (and all subspaces) is a metrizable space. Because convergence of sections in this norm implies local uniform convergence, the spaces of continuous, C^∞ and holomorphic functions are complete spaces for this norm, i.e., if s_j is a row of sections for which $\lim_{j\to\infty} ||s - s_j|| = 0$ for a section s, then s is continuous, C^∞ or holomorphic if the s_j are continuous, C^∞ or holomorphic respectively.

- We can now look at $B = \bar{B}(0, 1)$ which is the closure of $B(0, 1) = \{ s \in \mathcal{O}(M, V) \mid ||s|| < 1 \}$. B is a closed and bounded subset of the space of the holomorphic sections. We will prove that every sequence in B has a convergent subsequence. Because B is closed, this means that B is sequentially compact. Now every metrizable sequentially compact space is compact (see [7], paragraph 3.7), so the space B is also compact. Because B is compact, the space $\mathcal{O}(M, V)$ has to be finite-dimensional.

- We still have to prove that a sequence $(f_j)_{j=1}^{\infty}$ in B has a convergent subsequence. For each point $x \in M$ we again choose a chart (U_x, κ_x) such that there is a trivialisation ρ_x of the vector bundle. For $\tilde{U}_x = \kappa_x(U_x)$ we choose choose $r_x > 0$ such that $\tilde{V}_x = \bar{B}(y, r_x) \subset \tilde{U}_x, y = \kappa_x(x)$ and $d(\mathbb{C}^m \setminus \tilde{U}_x, \tilde{V}_x) = \eta_x > 0$. Take $\tilde{W}_x \subset \tilde{V}_x$ to be $\tilde{W}_x = B(y, \frac{1}{2}r_x)$. The open sets $W_x = \kappa_x^{-1}(\tilde{W}_x)$ cover M. Because M is compact we can select a finite subcover $(W_x)_{x \in I}$.
- We have now constructed a covering $(W_x)_{x \in I}$ which we will use to prove the theorem. A picture of this construction is given in figure 1.1. Before continuing the proof, we first consider why we need this complex construction. To find our covering $(W_x)_{x \in I}$ we needed the construction of the sets U_x, $\tilde{U}_x$, $\tilde{V}_x$ and $\tilde{W}_x$ for $x \in I$.
  - The steps from (U_x) to $(\tilde{U}_x)$ and from $(\tilde{W}_x)$ back to (W_x) are needed because we need a trivilisation of the vector bundle. We need this trivialization to be able to work with functions from $\mathbb{C}^m$ to $\mathbb{C}^n$, which are easier than sections of a vector bundle.
  - To be able to use Ascoli's theorem we need an equicontinous set of functions. We will use lemma 1.3.2 to prove the equicontinuity and therefore we need functions defined a compact set. For this we use the covering $(\tilde{V}_x)$ which are compact sets.
  - Finally we need to select a finite subcovering, so we select the subsets $\tilde{W}_x$ of $\tilde{V}_x$ which are open.

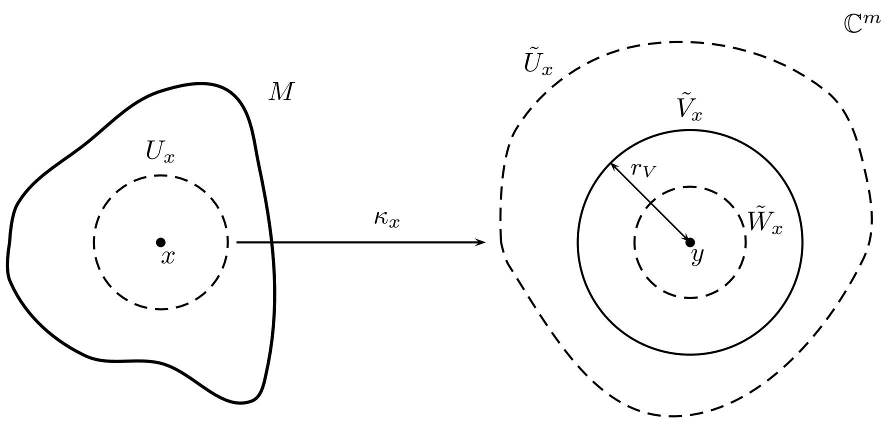

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Figure 1.1: Picture of the sets U_x, $\tilde{U}_x$, $\tilde{V}_x$ and $\tilde{W}_x$.

- From the covering $(W_x)_{x \in I}$ we obtain a finite covering $(V_x)_{x \in I}$ of M by compact subsets. The functions $g_{x,j} = \rho_x \circ f_j \circ \kappa_x^{-1}$ are a sequence of holomorphic functions from $\tilde{U}_x \subset \mathbb{C}^m$ to $\mathbb{C}^n$ for every $x \in I$. For every $x \in I$ the functions $(g_{x,j})_{j=1}^{\infty}$ are equicontinuous when restricted to $\tilde{V}_x \subset \tilde{U}_x$ by lemma 1.3.2.

  Because I is finite, we can take $I = \{ x_1, x_2, \dots, x_d \}$. By Ascoli's theorem the $g_{x_1,j} = \rho_{x_1} \circ f_j \circ \kappa_{x_1}^{-1}$ have a convergent subsequence $(g_{x_1,s_{1,j}})_{j=1}^{\infty}$. Now $(g_{x_2,s_{1,j}})_{j=1}^{\infty}$ is an equicontinuous sequence of functions on V_{x_2}. Therefore we can again select a subsequence $(g_{x_2,s_{2,j}})_{j=1}^{\infty}$ which is convergent on V_{x_2}. We can repeat this proces for every element of I. In this way we get sequences $(g_{x_k,s_{k,j}})_{j=1}^{\infty}$ that are convergent on V_{x_k}. The last sequence $(s_{n,j})_{j=1}^{\infty}$ gives a subsequence of the f_j, which we write as $h_j = f_{s_{n,j}}$.

- Now the subsequence $(h_j)_{j=1}^{\infty}$ converges on each V_x for $x \in I$, so there is a section h of the vector bundle V such that $h = \lim_{j\to\infty} h_j$. This means that $(\rho_x \circ h_j \circ \kappa_x^{-1})_{j=1}^{\infty}$ converges on each $\tilde{V}_x$ to $\rho_x \circ h \circ \kappa_x^{-1}$ and because $\tilde{V}_x$ is compact this convergence is uniform (on each $\tilde{V}_x$). From theorem 1.2.2 it follows that the limit function $\rho_x \circ h \circ \kappa_x^{-1}$ is holomorphic on each set $\tilde{V}_x$. As $\rho_x \circ h \circ \kappa_x^{-1}$ is holomorphic on each $\tilde{V}_x$, we see that h is holomorphic on each V_x and thus h is holomorphic on the entire space M.

□

1.4 Trivial vector bundles

A trival vector bundle (V, π) over M is a vector bundle for which $V = M \times \mathbb{C}^n$ and $\pi : V \to M : (m, v) \to m$. The holomorphic sections of V can be identified with the holomorphic functions from M to $\mathbb{C}^n$.

Theorem 1.4.1. *Let M be a compact, connected, complex manifold and f a holomorphic function on M. Then f is a constant function.*

Proof. The function f is continuous, so $|f|$ is a continuous function on M. Because M is compact, $|f|$ attains its maximum in some point x of M. Choose a chart (U, κ) with $x \in U$. Then $g = f \circ \kappa^{-1}$ is a holomorphic function on $V = \kappa(U) \subset \mathbb{C}^m$. Define for $\mathbf{z} \in \mathbb{C}^m$ the function $g_{\mathbf{z}} : t \to g(\kappa(x)+t\mathbf{z})$. The holomorphic function $|g_{\mathbf{z}}|$ has a maximum at $y = \kappa(x)$. From one-variable complex function theory we know that $g_{\mathbf{z}}$ is constant on V. (See for example [2, paragraph 5.4]). This means that g is constant on a neighbourhood of y, and therefore f is constant on a neighbourhood of x. Because this is true for every $x \in M$ and M is connected the function f is constant on M. □

Theorem 1.4.2. *Let (V, π) be a trivial vector bundle of rank r over the compact manifold M. Let d be the number of distinct connected components of M.*

Then $\mathcal{O}(M, V) \cong \mathbb{C}^{rd}$. The holomorphic functions are constant on each connected component of M.

Proof. Suppose $s \in \mathcal{O}(M, V)$. Then the coordinate functions $s_1, \ldots, s_n$ of s as a holomorphic function from M to $\mathbb{C}^n$ are holomorphic functions on M. By theorem 1.4.1 these functions are constant on each connected component of M, so s is constant on each connected component.

Let $M_1, \ldots, M_d$ be the connected components of M. Because every holomorphic section s is constant on each M_j we can write $s = \lambda_1 \mathrm{I}_{M_1} + \ldots + \lambda_d \mathrm{I}_{M_d}$ for unique $\lambda_j \in \mathbb{C}^r$ (here I_{M_j} is the indicator function of M_j). Now $\psi : \mathcal{O}(M, V) \to \mathbb{C}^{rd}$ defined by

$$\psi : s = \sum_{j=1}^{d} \lambda_j \mathrm{I}_{M_j} \to \lambda_1 \times \lambda_2 \times \cdots \times \lambda_d$$

is a linear bijective mapping and thus an isomorphism between $\mathcal{O}(M, V)$ and $\mathbb{C}^{rd}$. □

The last theorem gives a complete classification of the holomorphic sections of trivial vector bundles. In the next subsection we will see some examples of this.

1.5 Examples

In this section we will look at two vector bundles over the compact complex manifold $\mathbb{P}^1(\mathbb{C})$. First we will define $\mathbb{P}^1(\mathbb{C})$ and prove some of its properties. After that we will construct two vector bundles over $\mathbb{P}^1(\mathbb{C})$ and look at the holomorphic sections of these bundles.

Definition 1.5.1. Let $X_n = \mathbb{C}^{n+1} \setminus \{0\}$. On X_n we define the equivalence relation $: x \sim y \iff \exists c \in \mathbb{C} : c \cdot x = y$.

This defines an equivalence relation and we can construct the space $\mathbb{P}^n(\mathbb{C}) = X/\sim$. We denote the natural projection from X_n to $\mathbb{P}^n$ by π. For $x = (x_1, \ldots, x_{n+1})$ in X_n we write $[x] = [x_1 : x_2 : \ldots : x_{n+1}] = \pi(x) \in \mathbb{P}^n$.

The space $\mathbb{P}^n(\mathbb{C})$ is called the complex projective space (of dimension n). We can turn $\mathbb{P}^n$ into a complex n-dimensional manifold by providing it with the following charts. For $1 \le j \le n+1$ we define

$$\begin{aligned} &U_j = \{\, [x_1 : x_2 : \ldots : x_{n+1}] \in \mathbb{P}^n \mid x_j \neq 0 \,\}, \\ &\kappa_j : \mathbb{P}^n \to \mathbb{C}^n : [x_1 : x_2 : \ldots : x_{n+1}] \to \frac{1}{x_j}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n+1}). \end{aligned}$$

The U_j form an open cover of $\mathbb{P}^n$. Note that the definition of κ_j does not depend on the choice of x for the representant $[x]$. If $p = (p_1, \ldots, p_{n+1}) \in U_j \cap U_k \subset \mathbb{P}^n$ then $q = \kappa_j(p) = (q_1, \ldots, q_n) \in \kappa_j(U_j \cap U_k)$. If $k < j$ then $q_k \neq 0$ and if $K > j$ then $q_{k-1} \neq 0$. The coordinate transformations are

$$\begin{aligned} \kappa_j^{-1}(q) &= \kappa_j^{-1}((q_1, \ldots, q_n)) = [q_1, q_2, \ldots, q_{j-1}, 1, q_j, \ldots, q_n], \\ (\kappa_k \circ \kappa_j^{-1})(q) &= \begin{cases} \frac{1}{q_k}(q_1, q_2, \ldots, q_{k-1}, q_{k+1}, \ldots, q_{j-1}, 1, q_j, \ldots, q_n) & \text{if } k < j \\ \frac{1}{q_{k-1}}(q_1, q_2, \ldots, q_{j-1}, 1, q_j, \ldots, q_{k-2}, q_k, \ldots, q_n) & \text{if } k > j \end{cases}. \end{aligned}$$

These coordinate transformations are are complex-analytic.

The last thing we need to do to prove that $\mathbb{P}^n$ is a complex manifold, is to show that $\mathbb{P}^n$ is Hausdorff. Suppose $p = [x], q = [y] \in \mathbb{P}^n$ and $p \neq q$.

- If $p, q \in U_j$ we can select disjoint environments U_x and U_y of x and y respectively (because $\mathbb{C}^{n+1}$ is Hausdorff). Then $\pi(U_x)$ and $\pi(U_y)$ are disjoint open neighbourhoods of p and q.
- If there is no j for which $p, q \in U_j$, then for every $1 \le j \le n+1$ we have $x_j y_j = 0$. After a permutation of the indices we can arrange that $x = (x_1, x_2, \ldots, x_s, 0, \ldots, 0)$ and $y = (0, \ldots, 0, y_{s+1}, \ldots, y_{n+1})$. Take $U_x = \{\, z \in \mathbb{C}^{n+1} \mid |z_{n+1}| < |z_1| \,\}$ and $U_y = \{\, z \in \mathbb{C}^{n+1} \mid |z_1| < |z_{n+1}| \,\}$. Again $\pi(U_x)$ and $\pi(U_y)$ are disjoint neighbourhoods of p and q.

So $\mathbb{P}^n(\mathbb{C})$ is Hausdorff.

Note that the topology defined here, corresponds with the quotient topology for $\mathbb{P}^n$ as $\mathbb{P}^n = X/\sim$. (The quotient topology is defined as the unique topology for which the projection mapping $\pi : X \to \mathbb{P}^n$ is continuous and open, see [7, paragraph 2.11]).

Theorem 1.5.1. *The space* $\mathbb{P}^n(\mathbb{C})$ *is compact.*

Proof. $S^n = \{\, \mathbf{z} \in \mathbb{C}^{n+1} \mid |\mathbf{z}| = 1 \,\}$ is a compact subset of $\mathbb{C}^{n+1}$. For each $\mathbf{z} \in \mathbb{C}^{n+1} \setminus \{0\}$ there is a point $\mathbf{z}_0 = \mathbf{z}/|\mathbf{z}| \in S^n$ with $\pi(\mathbf{z}) = \pi(\mathbf{z}_0)$, so the projection $\pi|_{S^n} : S^n \to \mathbb{P}^n$ is surjective. The image of the compact set S^n under the continuous mapping π is $\mathbb{P}^n$, so $\mathbb{P}^n$ must be compact. □

For $n = 1$ we have that the space $\mathbb{P}^1$ is a complex one-dimensional manifold. There are two charts:

$$\begin{aligned} U_\infty &= \{\, [x : y] \in P^1 \mid x \neq 0 \,\}, \qquad \kappa_\infty : U_\infty \to \mathbb{C} : [1 : z] \to z \\ U_0 &= \{\, [x : y] \in P^1 \mid y \neq 0 \,\}, \qquad \kappa_0 : U_0 \to \mathbb{C} : [z : 1] \to z. \end{aligned}$$

There are two coordinate transformations. We have $\kappa_0(U_0 \cap U_\infty) = \kappa_\infty(U_0 \cap U_\infty) = \mathbb{C}^*$:

$$\kappa_\infty \circ \kappa_0^{-1} : \mathbb{C}^* \to \mathbb{C}^* : z \to \kappa_\infty([z:1]) = \frac{1}{z},$$
$$\kappa_0 \circ \kappa_\infty^{-1} : \mathbb{C}^* \to \mathbb{C}^* : z \to \frac{1}{z}.$$

Example 1. The manifold $\mathbb{P}^1$ is a compact. We can construct a trivial vector bundle of rank 1 over $\mathbb{P}^1$ by taking $E = \mathbb{P}^1 \times \mathbb{C}$ and $\pi : E \to \mathbb{P}^1 : (p, c) \to p$.

We will see that the space $\mathcal{O}(\mathbb{P}^1, E)$ is isomorphic to the constant functions on $\mathbb{P}^1$. Suppose f is a holomorphic function on $\mathbb{P}^1$. Then $f \circ \kappa_0^{-1}$ is an entire function on $\mathbb{C}$. Because $\lim_{|z|\to\infty}(f \circ \kappa_0^{-1})(z) = f(\infty) < \infty$, the function $f \circ \kappa_0^{-1}$ is bounded. By Liouville's theorem every bounded entire function is constant, so f must be a constant function.

This proves that $\mathcal{O}(\mathbb{P}^1, E)$ is equal to the collection of constant functions on $\mathbb{P}^1$ (which is a 1-dimensional space). This is in correspondence with theorem 1.4.2.

Example 2. Let $X = \mathbb{C}^2 \setminus \{0\}$, $H = \mathbb{C}^*$ and $V = \mathbb{C}_n$, $n \in \mathbb{Z}_{\geq 0}$. Here $\mathbb{C}_n$ is just $\mathbb{C}$ as a H-module under the action

$$\xi_n(h) : v \to \xi_n(h)v = h^{-n}v$$

for $v \in V$ and $h \in H$. This action gives an action on $X \times V$:

$$h \cdot (x, v) = (xh^{-1}, \xi_n(h)v).$$

This action is proper and free, so we can construct the manifold $L_n = X \times_H V = (X \times V)/H$. For the image of $(x, v) \in X \times V$ in $X \times_H V$ we write $[x, v]$. The projection $\pi : X \times_H V \to X/H : [x, v] \to xH$ gives a vector bundle called the vector bundle associated to the action ξ_n. In the next section we will look closer at the construction of this vector bundle, for now we will assume everything is properly defined.

$$\begin{array}{c} L_n = X \times_H V \\ \Big\downarrow \pi \\ X/H \cong \mathbb{P}^1(\mathbb{C}) \end{array}$$

The space of holomorphic sections $\mathcal{O}(L_n)$ is isomorphic to

$$\mathcal{O}(X, \mathbb{C}, \xi_n) = \{\, f : X \to \mathbb{C} \mid f \text{ is holomorphic}, f(xh) = h^n f(x) \,\}$$

under the mapping $\mathcal{O}(X, \mathbb{C}, \xi_n) \to \mathcal{O}(L_n) : f \to s_f$ where $s_f(xH) = [x, f(x)]$. The section s_f is well-defined because of the transformation properties of f.

We will prove that $\mathcal{O}(X, \mathbb{C}, \xi_n)$ is equal to the space $P_n(\mathbb{C}^2)$ of homogeneous polynomials of order n. This implies that $\dim \mathcal{O}(L_n) < \infty$ which also follows from theorem 1.3.3.

First note that $\mathcal{O}(X, \mathbb{C}, \xi_n) \cong \mathcal{O}(\mathbb{C}^2, \mathbb{C}, \xi_n)$ because every holomorphic function of two variables with an isolated singularity can be uniquely extended to a holomorphic function on the singular point (see [3]). In this case is is easy to see that the transformation property guarantees that for every $f \in \mathcal{O}(X, \mathbb{C}, \xi_n)$ we have $\lim_{z\to 0} f(z) = 0$ if $n > 0$ and $\lim_{z\to 0} f(z) < \infty$ if $n = 0$. Next notice that the transformation property together with theorem 1.2.3 gives that f is a polynomial in the two variables z_1 and z_2. It is also clear that every homogeneous polynomial in two coordinates factorizes to a function on $\mathbb{P}^1(\mathbb{C})$ which is holomorphic.

We know that the space $\mathcal{O}(\mathbb{C}^2, \xi) = P_n(\mathbb{C})$ has a basis of the polynomials $p_j = z_1^j z_2^{n-j}$ for $0 \leq j \leq n$. This gives that $\dim L_n = n + 1$ and the sections in $\mathcal{O}(L_n)$ have a basis q_j with

$$q_j([z_1 : z_2]) = [[z_1 : z_2], p_j(z_1, z_2)].$$

Again we see that this is well defined because of the transformation properties of p_j.

Chapter 2

Induced representations

Let G be a Lie group and H a closed subgroup of G. A representation π of G in V gives a representation of H in V by restricting π to H. For a representation ξ of H in V, there is no natural way to create a representation of G in V. But it is possible to create representations of G in function spaces in a systematic way from representations of H. One way to do this is by using *induced representations* which will be defined in this chapter.

2.1 Associated fiber bundles

Let X and Y be manifolds and H a Lie group with smooth actions α_1 and α_2 on X and Y. We will denote these actions by

$$\alpha_1(h,x) = x \cdot h^{-1}, \qquad \alpha_2(h,y) = h \cdot y.$$

We assume that the action from H on X is proper and free. Then X/H is a manifold and X is a principal fiber bundle with structure group H.

We can define an action α from H on the product $X \times Y$ by

$$\alpha(h,(x,y)) = (x,y)h = (x \cdot h^{-1}, h \cdot y).$$

This action is free, because H acts freely on the first component. To prove that the action is proper, we use the following theorem (which is proved in [9]):

Theorem 2.1.1. *Let M be a topological space and H a Lie group with a continuous (right) action on M. Then the following conditions are equivalent*

i) The action of H on M is proper.

ii) For every pair of compact subsets $C_1, C_2 \subset M$ the set $H_{C_1,C_2} = \{ h \in H \mid C_1 h \cap C_2 \neq 0 \}$ is compact.

Using this theorem we can prove that the action on $X \times Y$ is proper. Suppose A and B are compact subsets of $X \times Y$. The set $H_{A,B} = \{ h \in H \mid Ah \cap B \neq \emptyset \}$ is closed. Take compact subsets A_X, A_Y, B_X, B_Y such that $A \subset A_X \times A_Y$ and $B \subset B_X \times B_Y$. Now

$$\begin{aligned} H_{A,B} &= \{ h \in H \mid (A_X \times A_Y)h \cap (B_X \times B_Y) \neq \emptyset \} \\ &\subset \{ h \in H \mid A_X h \cap B_X \neq \emptyset \}. \end{aligned}$$

But this last set is just H_{A_X,B_X} and is compact because the action of H on X is proper. So $H_{A,B}$ is a closed subset of a compact set, implying that $H_{A,B}$ is compact. By the theorem the action α is proper.

Because α is a proper and free action, α is of principal fiber bundle type and the quotient space $X \times_H Y = (X \times Y)/H$ is a smooth manifold. The projection $\pi : X \times Y \to X$ induces a projection $\rho : X \times_H Y \to X/H$. We will write $[x, y] = (x, y)H$. The following diagram is commutative and $(X \times_H Y, \rho)$ is vector bundle over X/H with fiber Y.

$$\begin{array}{ccc} X \times Y & \longrightarrow & X \times_H Y \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \rho} \\ X & \longrightarrow & X/H \end{array}$$

For more details and a proof of the assertions here see [1, paragraph 2.4]. The fiber bundle $(X \times_H Y, \rho)$ is called the fiber bundle associated to the principal fiber bundle $X \to X/H$ using the action of H on Y.

We can consider the special case that Y is a finite-dimensional vector space. In this situation the fiber bundle $X \times_H Y$ is in fact a vector bundle. Another special case is the situation that $G = X$ is a Lie group, H is a closed subgroup of G and V is a finite-dimensional H-module (so V is a finite-dimensional vectorspace, and H has a representation ξ in V). In this case we have a free action from H on G by $h \cdot g = gh^{-1}$. This action is proper because the mapping $(g, h) \to (g, h \cdot g) = (g, gh^{-1})$ has a continuous inverse. H also has an action on V because V is an H module.

2.2 Induced representations

Let G be a Lie group, H a closed subgroup of G and ξ a finite dimensional representation of H in V. In the previous section we have seen that we can construct the vector bundle $\mathcal{V} = G \times_H V$ over G/H using the action

$$h(g, v) = (gh^{-1}, \xi(h)v).$$

We have a natural action from G on $G \times V$ defined by left multiplication on the first coordinate: $g \cdot (x, v) = (gx, v)$. Note that the action of H on $G \times V$ by restricting the action from G is very different from the action of H on $G \times V$. For this reason we will write from now on $h \cdot (x, v) = (hx, v)$ for the action of G (and H as a subgroup of G) on $G \times V$ and $h(x, v) = (xh^{-1}, \xi(h)^{-1}v)$ for the action of H on $G \times V$.

Because the actions of G and H on $G \times V$ commute, the action of G passes to the quotient $\mathcal{V} = G \times_H V$. Under this action $\mathcal{V}$ becomes a homogeneous vector bundle over G/H. We will write $h \cdot [x, v] = [hx, v]$ for the action of G (and H as a subgroup of G) on $\mathcal{V}$ and $h[x, v] = [xh^{-1}, \xi(h)^{-1}v]$ for the action of H on $\mathcal{V}$ (note that this last action is a trivial action, because $[xh^{-1}, \xi(h)^{-1}v] = [x, v]$).

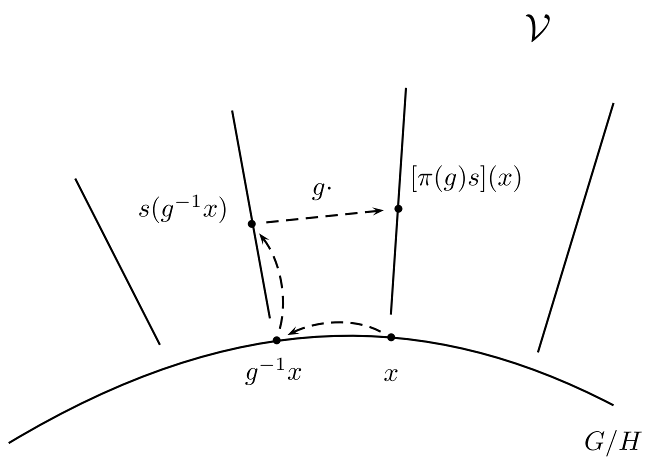
By $C(\mathcal{V})$, $C^\infty(\mathcal{V})$ and $\mathcal{O}(\mathcal{V})$ we denote the spaces of continuous, smooth and holomorphic sections of $\mathcal{V}$, respectively. G has a representation π in $C(\mathcal{V})$ defined by

$$[\pi(g)s](x) = g \cdot s(g^{-1}x) \tag{2.1}$$

for $g \in G$, $s \in C(\mathcal{V})$ and $x \in G/H$. It is easy to check that this defines a representation. Restriction of this representation gives representations of G in $C^\infty(\mathcal{V})$ and $\mathcal{O}(\mathcal{V})$. We call this representation of G in $C(\mathcal{V})$ the representation induced from ξ and write $\pi = \mathrm{ind}_H^G(\xi)$. A geometric picture of the construction of the section $\pi(g)s$ is given in picture 2.1.

2.3 Induced picture

We now have a representation of G in the space of sections $C(\mathcal{V})$. We will prove that this space is isomorphic to a subspace of the space of functions from G to V. This isomorphism gives a



)(0,0)(9,7)

Figure 2.1: The induced representation of ξ

representation of G in a subspace of $C(G, V)$. This realization of the induced representation of G is called the induced picture.

First we identify V with the fiber $\pi^{-1}(eH)$ through the mapping $v \to (e, v)$. Now for every section $s \in C(\mathcal{V})$ we can define the function $\phi_s : G \to V$ by

$$\phi_s(g) = g^{-1} \cdot s(gH). \tag{2.2}$$

Every function $\phi = \phi_s$ which we obtain in this way from a section has the following transformation property. If $s(gH) = [g, v]$ (so $\phi(g) = v$), then we have for $h \in H$ and $\phi = \phi_s$

$$\begin{aligned}
\phi(gh) &= (gh)^{-1} \cdot s(ghH) = h^{-1}g^{-1} \cdot s(gH) \\
&= h^{-1}g^{-1} \cdot [g, v] = [h^{-1}, v] \\
&= [e, \xi(h)^{-1}v] = \xi(h)^{-1}v
\end{aligned}$$

so

$$\phi(gh) = \xi(h)^{-1}\phi(g). \tag{2.3}$$

We define $C(G, V, \xi)$ to be the subspace of functions $\phi \in C(G, V)$ which satisfy the relation (2.3). So we have a mapping $s \to \phi_s : C(\mathcal{V}) \to C(G, V, \xi)$. For each function $\phi \in C(G, V, \xi)$ there is a section of $\mathcal{V}$ defined by

$$s_\phi(gH) = [g, \phi(g)]. \tag{2.4}$$

Note that this a good definition because of the transformation properties of ϕ. This means that $s_\phi(gH)$ is independent of the element g chosen to represent gH. The mapping $\phi \to s_\phi$ is the inverse of $s \to \phi_s$. The spaces $C(\mathcal{V})$ and $C(G, \xi) = C(G, V, \xi)$ are isomorphic under this mapping.

Because $C(G,V)$ is isomorphic to $C(G) \otimes V$ by mapping $f \otimes v$ to the function $g \to f(g)v$, the space $C(G,\xi)$ is isomorphic to a subspace of $C(G) \otimes V$. The action $R \otimes \xi$ is an action of H on $C(G) \otimes V$ defined by

$$h \cdot (f, v) = (f \circ R_h, \xi(h)v)$$

where R_h means right multiplication with h. Now $C(G,\xi)$ is isomorphic to $(C(G) \otimes V)^H = \{ x \in C(G) \otimes V \mid hx = x \ \forall h \in H \}$.

With this identification of $C(\mathcal{V})$, $C(G,\xi)$ and $(C(G) \otimes V)^H$ we can look at the realization $\bar{\pi}$ of $\pi = \mathrm{ind}_H^G(\xi)$ on $C(G,\xi)$. One can easily check that $\bar{\pi}$ is given by

$$[\bar{\pi}(g)\phi](x) = \phi(g^{-1}x).$$

Indeed, if s is a section in $C(\mathcal{V})$, f is the function corresponding to s, $s' = \pi(g)s, g \in G$ and f' is the function corresponding to s' then we have for $x \in G$:

$$\begin{aligned} f'(x) &:= x^{-1} \cdot (\pi(g)s)(xH) \\ &= x^{-1} \cdot g \cdot s(g^{-1}xH) \\ &= (g^{-1}x)^{-1} s(g^{-1}xH) \\ &= f(g^{-1}x) \end{aligned}$$

so the induced action in $C(G,\mathbb{C},\xi)$ is indeed defined by left multiplication with the inverse element.

For every induced representation π we have found an equivalent representation $\bar{\pi}$. The realization $\bar{\pi}$ of π is called the "induced picture". From now on we will write $\pi = \mathrm{ind}_H^G(\xi)$ for both the action on $C(G,\xi)$ and on $C(\mathcal{V})$.

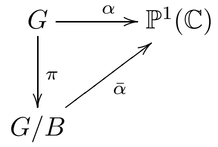
2.4 Example

In this example we look at the Lie group SL(2,$\mathbb{C}$) and a subgroup B which is the stabilizer of the action of SL(2,$\mathbb{C}$) on $\mathbb{P}^1(\mathbb{C})$. We construct the induced represention for several representations of B and show how this representation works. Finally we prove that the space of holomorphic sections is finite-dimensional (which also directly follows from theorem 1.3.3) and calculate the dimension of the space of holomorphic sections.

- Let $G = \mathrm{SL}(2,\mathbb{C})$. G has a natural action on $\mathbb{C}^2$ by multiplication. This action induces an action α from G on $\mathbb{P}^1(\mathbb{C})$ ($\mathbb{P}^1(\mathbb{C})$ is defined in section 1.5). This action is transitive on the element $p = [1:0]$ of $\mathbb{P}^1$. The stabilizer group of p is

$$\begin{aligned} B = G_p &= \{ g \in \mathrm{SL}(2,\mathbb{C}) \mid g \cdot p = p \} \\ &= \{ g \in \mathrm{SL}(2,\mathbb{C}) \mid g \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{C} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \} \\ &= \{ g \in G \mid (g)_{i1} = 0 \text{ for } i > 1 \}. \end{aligned}$$

So $B = \{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}) \}$. The mapping $\alpha : G \to \mathbb{P}^1 : g \to g \cdot p$ factorises over a bijection $\bar{\alpha}$ of G/B to $\mathbb{P}^1(\mathbb{C})$.



We find that under the mapping $\bar{\alpha}$

$$\begin{pmatrix} a & 0 \\ 1 & a^{-1} \end{pmatrix} B \to [a:1], \qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} B \to [0:1], \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} B \to [1:0].$$

- For every $n \in \mathbb{Z}$ the group B has a representation ξ_n in $\mathbb{C}$ defined by $(c \in \mathbb{C})$

$$\xi_n(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix})c = a^{-n}c.$$

  We have a Lie group G and a subgroup B, so in the same way as in section 2.1 we can define a free and proper action from B on the product SL(2,$\mathbb{C}$) $\times \mathbb{C}_n$ by

$$\zeta_n : (h, (g, c)) \to b \cdot (g, c) = (gh^{-1}, h^{-n}c).$$

  We can construct the vector bundle $\mathcal{V} = \mathrm{SL}(2,\mathbb{C}) \times_B \mathbb{C}$ and look at the induced representation $\pi = \pi_n = \mathrm{ind}_B^{\mathrm{SL}(2,\mathbb{C})}(\xi_n)$. This representation works on the space $C(G, \mathbb{C}, \xi)$ which is the space of continuous functions $G \to \mathbb{C}$ which satisfy the transformation rule $f(xh) = h^{-1} \cdot f(x) = a^n f(x)$ for $x \in G$, $h = \left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right) \in B$.

- To make this representation more concrete, we look at the function $f : G \to \mathbb{C} : \left(\begin{smallmatrix} x_1 & x_2 \\ x_3 & x_4 \end{smallmatrix}\right) \to x_1^n$. The function f clearly is an element in $C(G, \mathbb{C}, \xi_n)$. For every $g = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in$ SL(n,$\mathbb{C}$) we have the induced representation $\pi(g)$ working on f. This gives the function $p = \pi(g)f$:

$$\begin{aligned} p(\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}) &= [\pi(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix})f] \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \\ &= f(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{-1} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}) \\ &= f(\begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}) \\ &= (x_1\delta - x_3\beta)^n. \end{aligned}$$

  It is easy to check that p again satisfies the transformation rule (2.3), so $p \in C(G, \mathbb{C}, \xi_n)$.

Theorem 2.4.1. *Let $G = SL(2,\mathbb{C})$, B, ξ_n and $\mathcal{V} = G \times_B \mathbb{C}_n$ be defined as before. The space of holomorphic sections $\mathcal{O}(G/B, \mathcal{V})$ is*

- *empty if $n < 0$;*
- *isomorphic to the space of homogeneous polynomials of degree n in 2 variables if $n \geq 0$.*

Proof. Let s be a section in $\mathcal{O}(G/B, \mathcal{V})$ and let $f = f_s$ be the corresponding holomorphic function on G. For every $x = \left(\begin{smallmatrix} x_1 & x_2 \\ x_3 & x_4 \end{smallmatrix}\right) \in G$ and $h = \left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right) \in B$. we have the following transformation property

$$\begin{aligned} f(x) = f(\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}) &= \xi(h)f(xh) \\ &= a^{-n}f(\begin{pmatrix} ax_1 & bx_1 + a^{-1}x_2 \\ ax_3 & bx_3 + a^{-1}x_4 \end{pmatrix}) \\ &= a^{-n}f(xh) \end{aligned}$$

Because x is in SL(2,$\mathbb{C}$) either x_1 or x_3 is non-zero. We now assume x_1 is non-zero (the case $x_3 \neq 0$ is analogous). We can take $a = 1$ and $b = -\frac{x_2}{x_1}$ to get

$$f(x) = f(\begin{pmatrix} x_1 & 0 \\ x_3 & -\frac{x_2x_3}{x_1} + \frac{x_1x_4}{x_1} \end{pmatrix}) = f(\begin{pmatrix} x_1 & 0 \\ x_3 & {x_1}^{-1} \end{pmatrix}).$$

We see that f depends only on x_1 and x_3, so f factorizes to a function $\bar{f}$ of the projection of x of the variables x_1 and x_3. We denote the projection by π.

We can now take $b = 0$ and a arbitrary to get

$$\begin{aligned} f(x) &= a^{-n} f(\begin{pmatrix} ax_1 & 0 \\ ax_3 & a^{-1}{x_1}^{-1} \end{pmatrix}) = a^{-n} \bar{f}(\begin{pmatrix} ax_1 \\ ax_3 \end{pmatrix}) \\ &= a^{-n} \bar{f}(\pi(ax)) = a^{-n} f(ax) \\ &\Rightarrow f(ax) = a^n f(x). \end{aligned}$$

From theorem 1.2.3 it follows that f is a polynomial of degree n in x_1 and x_3 if $n \geq 0$ or zero if $n < 0$. □

Chapter 3

The Borel-Weil Theorem

The main goal of this section is to prove the Borel-Weil theorem. We have a complete classification of all irreducible representations of a Lie group G in terms of the (dominant) weights of its Lie algebra (see theorem 3.3.3). The Borel-Weil theorem gives a way to create the representation corresponding to a dominant weight.

Before we can prove the Borel-Weil theorem (for the group SL(n,$\mathbb{C}$)) we first look at complex groups, weights, characters and the Bruhat-decomposition of a group. This is done in the sections 3.2 to 3.5.

3.1 Construction of a representation with highest weight λ

We give a short sketch of the various steps used in the final proof of the Borel-Weil theorem. The purpose of this sketch is giving the reader an idea how the theory from the next sections will be used. We start with a compact Lie group G and a dominant weight λ.

- First we choose a torus T in G and calculate the roots and root spaces of $\mathfrak{g}^{\mathbb{C}}$. Using the root-space decomposition we define a subgroup B of the complexification $G^{\mathbb{C}}$ of G corresponding to a subalgebra of $\mathfrak{g}^{\mathbb{C}}$.
- The weight λ is used to construct a character χ on $H = T^{\mathbb{C}}$. The character is then extended to a one-dimensional representation of the subgroup B.
- We can use the construction of induced representations to construct a representation π of G (or $G^{\mathbb{C}}$) using the subgroup B and representation χ.
- This induced representation has all the properties we need: it is irreducible and has highest weight λ. To prove these properties we need some more theory from the sections 3.2 to 3.5.

3.2 Complexification of groups

To prove the general Borel-Weil theorem, we need the complexification of a compact Lie group. The reason for this is the following: in the proof of the Borel-Weil theorem we need to prove that the representation space of the induced representation (the space of holomorphic sections) is finite-dimensional. For this we use theorem 1.3.3 so we need to look at the holomorphic sections of a compact complex manifold. For this we use a compact complex manifold defined by taking a quotient of the complexification of G and a suitable subgroup. For this reason, we will first look at some results on the relations between complex and real groups (and algebras).

First we suppose $G^{\mathbb{C}}$ is a complex Lie group with Lie algebra $\mathfrak{g}^{\mathbb{C}}$. A real Lie subalgebra $\mathfrak{u}_0$ is called a real form for $\mathfrak{g}^{\mathbb{C}}$ if

$$\left(\mathfrak{g}^{\mathbb{C}}\right)^{\mathbb{R}} = \mathfrak{u}_0 \oplus_{\mathbb{R}} i\mathfrak{u}_0.$$

Here $(\mathfrak{g}^{\mathbb{C}})^{\mathbb{R}}$ is $\mathfrak{g}^{\mathbb{C}}$ as a real vector space. It is possible to show that every complex semi-simple group has a compact real form $\mathfrak{u}_0$. The connected subgroup U_0 of G generated by $\mathfrak{u}_0$ is a closed compact subgroup of G.

Now suppose that G is a connected compact real Lie group with Lie algebra $\mathfrak{g}$. We can define the complexification of $\mathfrak{g}$ by

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}.$$

This means that $(\mathfrak{g}^{\mathbb{C}})^{\mathbb{R}} = \mathfrak{g} \oplus_{\mathbb{R}} i\mathfrak{g}$. Now we want to define the complexification of G. Because G is compact and connected, G is a reductive Lie group and we can embed G as a closed subgroup of $\mathrm{GL}(V)$ for some real vector space V. This gives an embedding of $\mathfrak{g}$ and $\mathfrak{g}^{\mathbb{C}}$ as a subalgebra of $\mathrm{End}(V)$ and $\mathrm{End}(V^{\mathbb{C}})$ respectively (here $V^{\mathbb{C}} = V \oplus iV$). We can define $G^{\mathbb{C}}$ as the connected subgroup of $\mathrm{GL}(V^{\mathbb{C}})$ with Lie algebra $\mathfrak{g}^{\mathbb{C}}$. The group $G^{\mathbb{C}}$ is called the complexification of G. It can be shown that up to isomorphisms the complexification $G^{\mathbb{C}}$ is unique. With this definition G is embedded as a closed subgroup of $G^{\mathbb{C}}$.

Example 3. Take $G = \mathrm{SL}(2,\mathbb{C})$. On the Lie algebra level we have $\mathrm{sl}(2,\mathbb{C}) = \mathbb{C}X \oplus \mathbb{C}H \oplus \mathbb{C}Y$, where

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

We can take $\mathfrak{u}_0 = i\mathbb{R}(X+Y) + i\mathbb{R}H + \mathbb{R}(X-Y)$. Then the subgroup K generated by $\mathfrak{u}_0$ is equal to $\mathrm{SU}(2)$ which is a compact connected subgroup of $\mathrm{SL}(2,\mathbb{C})$.

We need two theorems on the relation between a compact Lie group G and its complexification $G^{\mathbb{C}}$.

Theorem 3.2.1. *Suppose G is a compact connected Lie group which can be embedded in $GL(V)$. Define $G^{\mathbb{C}}$ and $\mathfrak{g}^{\mathbb{C}}$ as before. Let T be a maximal torus in G with Lie algebra $\mathfrak{t}$.*

Then $\mathfrak{h} = \mathfrak{t} \oplus i\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. Let B be connected subgroup of $G^{\mathbb{C}}$ with Lie algebra $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha>0} \mathfrak{g}_\alpha$.

Then $T \subset B$ and the mapping $G/T \to G^{\mathbb{C}}/B : gT \to gB$ is a real-analytic diffeomorphism. Because G is already compact, the spaces G/T and $G^{\mathbb{C}}/B$ are compact.

Proof. See [1] on page 297. □

Theorem 3.2.2 (Weyl's unitary trick)**.** *Let G be a simply connected linear reductive Lie group and let $G^{\mathbb{C}}$ be its complexification. Let $\mathfrak{g} = \mathfrak{l} + \mathfrak{p}$ be the Cartan decomposition of the Lie algebra $\mathfrak{g}$. Let U and $G^{\mathbb{C}}$ be the connected analytic groups of matrices with Lie algebras $\mathfrak{u} = \mathfrak{l} + i\mathfrak{p}$ and $\mathfrak{g}^{\mathbb{C}} = (\mathfrak{l} + \mathfrak{p})^{\mathbb{C}}$.*

If V is a finite-dimensional complex vector space, then a representation of any of the following kinds leads, via the formula $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$, to a representation of each of the other kinds. Under this correspondance invariant subspaces and equivalences are preserved:

(a) a representation of G on V

(b) a representation of U on V

(c) a holomorphic representation of $G^{\mathbb{C}}$ on V

(d) a representation of $\mathfrak{g}$ on V

(e) a representation of $\mathfrak{u}$ on V

(f) a complex-linear representation of $\mathfrak{g}^{\mathbb{C}}$ on V

Proof. See Proposition 5.7 in [4]. We will need only the equivalence of (a) and (c). Using this theorem we can switch between representations of a simply connected compact Lie group and its complexification. □

3.3 Weights and characters

In the entire section 3.3 we assume G is a compact connected Lie group with Lie algebra $\mathfrak{g}$ and a maximal torus T with corresponding Lie algebra $\mathfrak{t}$.

Definition 3.3.1. Let (π, V) be a finite-dimensional representation of a group G. We define the character of π to be the function $\chi_\pi : g \to \operatorname{tr} \pi(g)$.

Every character is a continuous, conjugacy-invariant function on G. If π is a non-trivial irreducible finite-dimensional representation of T in V, π must be one-dimensional. For $x \in T$, $\pi(x)$ acts on V by multiplication with the element $\chi_\pi(x) = \operatorname{tr} \pi(x)$. The mapping $\chi_\pi : T \to \mathbb{C}^*$ is in fact a 1-dimensional representation of T in the vector space $\mathbb{C}$.

3.3.1 Weights

Definition 3.3.2. We define $\Lambda = \ker(\exp|_{\mathfrak{t}}) = \{ X \in \mathfrak{g} \mid \exp X = e \}$. Λ is called the T-lattice.

We define the weights of T to be the set $\{ \mu \in i\mathfrak{t}^* \mid \mu(\Lambda) \in 2\pi i\mathbb{Z} \}$. We denote the set of weights of T by $\hat{T}$.

The notation $\hat{G}$ is also used for the set of equivalence classes of irreducible representations of a Lie group G. The following theorem says that these sets are isomorphic, so we can use the same notation for weights and equivalence classes of irreducible representations.

Theorem 3.3.1. *The class $\hat{T}$ of irreducible representations of T is in bijective correspondance with the set of weights of T.*

Proof. The irreducible representations of T are all one-dimensional (because T is abelian), so every irreducible representation is equivalent to a character of T.

If μ is a weight of $\mathfrak{t}$, we can define a character on T by

$$\chi_\mu : t \in T \to t^\mu = e^{\mu(X)}$$

if $t = \exp X$. This is a well defined character, because if $t = \exp X = \exp X'$ then $e = \exp(X - X')$ so $X - X' \in \Lambda$. Because μ is a weight, in follows that $e^{\mu(X-X')} = 1$. Now

$$\begin{aligned} e^{\mu(X)} &= e^{\mu(X'+X-X')} = e^{\mu(X')} e^{\mu(X-X')} \\ &= e^{\mu(X')}. \end{aligned}$$

Also note that for every $t \in T$ there is a $X \in \mathfrak{t}$ such that $t = \exp X$, because T is connected and abelian.

For every character χ of T we can define $\mu = T_e\chi : \mathfrak{t} \to \mathbb{C}$. The following diagram is commutative.

$$\begin{array}{ccc} T & \xrightarrow{\chi} & \mathbb{C}^* \\ \uparrow{\scriptstyle \exp} & & \uparrow{\scriptstyle e^{\cdot}} \\ \mathfrak{t} & \xrightarrow{\mu} & \mathbb{C} \end{array}$$

We first prove that μ is in fact a map $\mathfrak{t} \to i\mathbb{R}$. We know T is compact because G is compact, so the image $\chi(T)$ is a compact subset of $\mathbb{C}$. Now suppose $X \in \mathfrak{t}$ such that $\mu(X) = a + ib$ with $a, b \in \mathbb{R}$. Then we have that $A = \{ \chi(\exp(nX)) \mid n \in \mathbb{Z} \}$ is a subset of $\chi(T)$ and this implies that A is bounded. On the other hand $A = \{ e^{\mu(nX)} \mid n \in Z \} = \{ e^{na} e^{inb} \mid n \in \mathbb{Z} \}$. Because A is bounded, it follows that $a = 0$. This means that μ maps $\mathfrak{t}$ into $i\mathbb{R}$.

From the diagram we also see that $e^{\mu(\Lambda)} = (\chi \circ \exp)(\Lambda) = \chi(0) = 0$. This implies $\mu(\Lambda) \subset 2\pi i\mathbb{Z}$, so μ is a weight of $\mathfrak{t}$. From the diagram it also follows that χ and the mapping $t \to t^\mu = t^{T_e\chi}$ are equal. □

There is also a definition of the weights of a representation π of G in V in terms of the weight spaces of π. We will now show the relation between these definitions of weights.

If π is a representation of G we can define for every linear form $\mu : \mathfrak{t} \to i\mathbb{R}$ the corresponding weight space

$$V_\mu = \{ v \in V \mid \pi(X)v = \mu(X)v, \ \forall X \in \mathfrak{t} \}.$$

If $V_\mu \neq 0$ we call μ a weight of π. Now if $V_\mu \neq 0$ then $\pi(t)$, with $t = \exp X \in T$, acts on V_μ as multiplication by $e^{\mu(X)} = t^\mu$ because

$$\begin{aligned}\pi(t) : v \to & \pi(\exp X)v = \exp(\pi_* X)v \\ & = e^{\mu(X)}v.\end{aligned}$$

So $\pi(t)$ acts on the space V_μ as multiplication by t^μ and this defines a character on T. This character corresponds to the weight μ in the original definition of weights. So we see that a weight of the representation π of G is also a weight of the Lie algebra $\mathfrak{t}$.

From now on we will assume that we have chosen a set of positive roots P corresponding to a Weyl chamber $\mathfrak{c}$.

Definition 3.3.3. A weight λ of a representation π is called a highest weight for that representation if one of the following equivalent conditions is satisfied:

(i) If $\alpha \in P$, then $\lambda + \alpha$ is not a weight of π.

(ii) For every weight μ of π we have $\mu = \lambda - \sum_{\alpha \in P} n_\alpha \alpha$ for $n_\alpha \in \mathbb{Z}_{\leq 0}$.

The proof of the equivalence of these conditions can be found in [1]. These highest weights characterize a representation in the following sense:

Theorem 3.3.2. *Let G be a compact connected Lie group and let π and π' be two irreducible representations of G with highest weights λ and λ' respectively. Then π is equivalent π' if and only if $\lambda = \lambda'$.*

Proof. This follows from Corallary 4.11.5 from [1] and Lemma 20.2 from [9]. □

3.3.2 Dominant weights

We have seen that the weights of a representation are also weights of the torus T. For an irreducible representation there is the highest weight characterizing the representation. For the weights of a torus there is a similar concept: the dominant weights. Recall that for every root α there is a unique element $\alpha^\vee$ in $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ such that $\alpha(\alpha^\vee) = 2$. The elements $\alpha^\vee$ are called the co-roots.

Definition 3.3.4. A weight μ of the Lie algebra $\mathfrak{t}$ is called dominant (with respect to a positive root system P) if $\mu(\alpha^\vee) \geq 0$ for all $\alpha \in P$.

To prove the Borel-Weil theorem at the end of this section, we need the following theorem. The proof can be found in [1, paragraph 4.9]. It says that every dominant weight is the highest weight for some representation.

Theorem 3.3.3 (Highest Weight Theorem)**.** *Let G be a compact connected Lie group. The mapping which assings to each irreducible representation of G its highest weight is a bijection from $\hat{G}$ to the set of dominant weights in $\hat{T}$.*

In particular: every weight of a representation is a weight of the torus $\mathfrak{t}$; every highest weight of a representation is a dominant weight.

If G is a simply connected Lie group the situation is a bit more simple. After a choice of simple roots $S = \{\alpha_i, \ldots, \alpha_n\}$ (so S is a fundamental system) we can define the fundamental weights ω_i by

$$2\omega_i(\alpha_j^\vee)/|\alpha|^2 = \delta_{ij}. \tag{3.1}$$

The fundamental weights are indeed weights as in definition 3.3.4. Note that every fundamental weight is a dominant weight. In fact every dominant weight is the (positive) sum of fundamental weights. This is in general not true if G is not simply connected.

Example 4. Take $G = T = \mathbb{R}/\mathbb{Z}$ (with addition as the group operation). The Lie algebra $\mathfrak{t}$ of T is equal to $\mathbb{R}$. For $X \in \mathfrak{t}$, the solution to $\frac{dh}{dt}(t) = v_X(h(t))$, $h(0) = e$ is given by $h(t) = Xt$. Here v_X is the left-invariant vector field defined by $v_X : G \to TG : g \to T_eL_g(X) = X$). So the exponential mapping $\mathfrak{t} \to T$ is given by $X \to X + \mathbb{Z}$.

The T-lattice is $\Lambda = \ker \exp = \mathbb{Z}$. The weights of $\mathfrak{t}$ are $\mu_n : X \to 2\pi i n X$, $n \in \mathbb{Z}$. For every weight μ_n the corresponding character on T is defined by

$$\chi_n : t \to t^{\mu_n} = e^{\mu_n(t)} = e^{2\pi i n t}.$$

So all irreducible representations of the circle T are given by the functions $t \to e^{2\pi i n t}$.

Note that the representation $\pi_n(t) = e^{2\pi i n t}$ has exaclty one weight μ_n, this weight is a highest weight for this representation. Because there are no roots in $\mathfrak{t}$, all weights of $\mathfrak{T}$ are dominant weights.

3.3.3 Extending characters

Suppose we have a compact connected Lie group G with maximal torus T. For the Lie algebra $\mathfrak{g}^{\mathbb{C}}$ we have the root space decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha.$$

After a choice of positive roots P we can define $\mathfrak{n} = \oplus_{\alpha \in P} \mathfrak{g}_\alpha$, $\bar{\mathfrak{n}} = \oplus_{\alpha \in P} \mathfrak{g}_{-\alpha}$, $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ and $\bar{\mathfrak{b}} = \mathfrak{h} \oplus \bar{\mathfrak{n}}$. Now $B^+ = N_{G^{\mathbb{C}}}(\mathfrak{b}) = \exp \mathfrak{b}$ and $B^- = N_{G^{\mathbb{C}}}(\bar{\mathfrak{b}}) = \exp \bar{\mathfrak{b}}$ are Lie subgroups of $G^{\mathbb{C}}$, and both B^+ and B^- are Borel subgroups (maximal solvable subgroups).

Lemma 3.3.4. *Every character χ of T can be uniquely extended a homomorphism $B^+ \to \mathbb{C}^*$ or to a homomorphism of $B^- \to \mathbb{C}^*$.*

Proof. The proof can be found in [1] or [8]. In the proof there is also a construction the extension of the character to B^+ or B^-. The construction follows from requiring $\chi(n) = \chi(e)$ for all $n \in N^+$ or N^-. □

3.3.4 Line bundles

Now that we can extend characters of T to representations of $B = B^-$, we can define line bundles (vector bundles of rank 1) over $G^{\mathbb{C}}/B$. For every weight λ (corresponding to the torus T of a Lie group G) define L_λ to be the line bundle

$$L_\lambda = G \times_B \mathbb{C} \tag{3.2}$$

where the representation ξ of B on $\mathbb{C}$ is defined by extending the character $t \to t^\lambda$.

3.4 Highest weight space of irreducible representations

Suppose we have a representation (π, V) of a Lie group G. If N is a subgroup of G then we write V^N for the set of fixed points of V under the action of N. So

$$V^N = \{\, v \in V \mid \pi(g)v = v, \text{ for all } g \in N \,\}.$$

In a similar way we can write

$$V^{\mathfrak{n}} = \{\, v \in V \mid \pi_*(X)v = 0, \text{ for all } X \in \mathfrak{n} \,\}$$

if $\mathfrak{n}$ is a Lie subalgebra of $\mathfrak{g}$ or $\mathfrak{g}^{\mathbb{C}}$.

For a connected Lie subgroup N, the sets V^N and $V^{\mathfrak{n}}$ are equal if $\mathfrak{n} = \mathrm{Lie}(N)$. We can prove this in two steps

- If $v \in V^N$, then for all $X \in \mathfrak{n}$ we have that $\exp tX \in N$ so $\pi(\exp tX)v = v$. From this it follows

  $$\pi_*(X)v = T_e\pi(X)v = \left.\frac{d}{dt}\right|_{t=0} \pi(\exp tX)v = \left.\frac{d}{dt}\right|_{t=0} v = 0$$

  and thus $X \in \mathfrak{n}$.

- If $v \in V^{\mathfrak{n}}$ then for all $X \in \mathfrak{n}$ we have $\pi_*(X)v = 0$. Now suppose $n \in N$. Because N is connected we have $N = N_e$ and we can write $n = \exp X_1 \exp X_2 \cdots \exp X_j$ for certain $X_1, \ldots, X_j \in \mathfrak{n}$.

  Now for every X_k we have for all t_0

  $$\begin{aligned}\left.\frac{d}{dt}\right|_{t=t_0} \pi(\exp tX_k) &= \left.\frac{d}{ds}\right|_{s=0} \pi(\exp(t_0+s)X_k) \\ &= \pi(\exp t_0X_k) \left.\frac{d}{ds}\right|_{s=0} \pi(\exp(s)X_k) \\ &= \pi(\exp t_0X_k)\pi_*(X_k)v = 0\end{aligned}$$

  so $\pi(\exp X_k)v = v$ for all $1 \leq k \leq j$. From this it follows that

  $$\begin{aligned}\pi(n)v &= \pi(\exp X_1 \exp X_2 \cdots \exp X_j)v \\ &= \pi(\exp X_1)\pi(\exp X_2)\cdots\pi(\exp X_j)v = v.\end{aligned}$$

  This means that $n \in V^N$.

Theorem 3.4.1. *Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$. After a choice of positive roots define $\mathfrak{n} = \oplus_{\alpha>0}\mathfrak{g}_\alpha$ and N as the subgroup generated by $\exp \mathfrak{n}$.*

Then a non-trivial representation π of G in a finite-dimensional vector space V is irreducible if and only if $\dim V^N = 1$.

Proof. The proof that for a non-trivial irreducible representation the dimension of the space V^N is one follows from Theorem 4.11.4 in [1] and the preceding remarks on V^N and $V^{\mathfrak{n}}$.

Now suppose π is not irreducible. Then V can be written as a direct sum of $\pi(G)$-invariant subspaces $V = \oplus_j V_j$ such that the representation π_j of π on each of the subspaces V_j is irreducible for each j (because G is compact). This means that for every subspace we have $\dim V_j^N = 1$ if $V_j \neq 0$. We can easily see that $V^N = \oplus_j V_j^N$. Because π is reducible we see that $\dim V^N = \sum_j \dim V_j^N > 1$. ∎

If π is irreducible, then $V^N = V_\lambda$ for a unique weight λ. This weight λ is the heighest weight of the representation π.

3.5 Orbit structure on $G^{\mathbb{C}}/B$

Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$. Choose a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ and define $T = \exp(\mathfrak{t})$.

The complexification $\mathfrak{g}^{\mathbb{C}}$ of the Lie algebra $\mathfrak{g}$ is the direct sum $\mathfrak{g}^{\mathbb{C}} = \mathfrak{h} \oplus \bigoplus_{\alpha\in R} \mathfrak{g}_\alpha$ where R is the collection of roots of $\mathfrak{g}^{\mathbb{C}}$. We define $\mathfrak{n} = \oplus_{\alpha>0}\mathfrak{g}_\alpha$, $\bar{\mathfrak{n}} = \oplus_{\alpha>0}\mathfrak{g}_{-\alpha}$ and $\mathfrak{h} = \mathfrak{t}^{\mathbb{C}} = \mathfrak{g}_0$. Now $\mathfrak{g}^{\mathbb{C}} = \bar{\mathfrak{n}} \oplus \mathfrak{h} \oplus \mathfrak{n}$. Take $N = N^+ = \exp \mathfrak{n}$ and $B = B^- = \exp(\mathfrak{h} \oplus \bar{\mathfrak{n}})$. B is a Lie subgroup of $G^{\mathbb{C}}$ and $G^{\mathbb{C}}/B$ has the structure of a compact complex manifold. The subgroup N has a natural action in $G^{\mathbb{C}}/B$ by multiplication on the left.

Theorem 3.5.1 (Bruhat-decomposition)**.** *In the above notation we have that the orbits of N in $G^{\mathbb{C}}/B$ are precisely the NwB for $w \in W$, where $W = N_{G^{\mathbb{C}}}(T)/T$ is the Weyl-group. The orbits are are immersed manifolds. This means that we have the decomposition*

$$G^{\mathbb{C}}/B = \coprod_{w\in W} NwB.$$

The orbit NeB is dense and open in $G^{\mathbb{C}}$. For all other orbits NwB ($w \neq e$) the dimension is strictly lower then $\dim G^{\mathbb{C}}$*.*

Proof. The decomposition is called Bruhat-decomposition. The theorem stated here is a special case of the Bruhat decomposition for real groups. The proof in this case can be found in [4] or [5]. □

Example 5. We look at $G^{\mathbb{C}} = \mathrm{SL}(2,\mathbb{C})$. We choose

$$H = \{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{C}^* \}$$

as a Cartan subgroup in $G^{\mathbb{C}}$. We have

$$B = \{ \begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix} \mid a \in \mathbb{C}^*, b \in \mathbb{C} \},$$
$$N = \{ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \mid c \in \mathbb{C} \}.$$

The centralizer of T in $G^{\mathbb{C}}$ is $Z_{G^{\mathbb{C}}}(T) = T$ and the normalizer of T equals

$$N_{G^{\mathbb{C}}}(T) = T \cup \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} T.$$

So W has two elements eT and wT where $w = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

The Bruhat-decomposition of SL(2,ℂ) is

$$\mathrm{SL}(2,\mathbb{C}) = NeB \coprod NwB$$

and $\dim NeB = 1$, $\dim NwB = 0$ in G/B.

We can explicitly show that this decomposition holds. Suppose $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$. If $d = 0$ then $g \in wB = NwB$. If $d \neq 0$ we can write

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d^{-1} + bcd^{-1} & b \\ c & d \end{pmatrix}$$
$$= \begin{pmatrix} 1 & d^{-1}b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d^{-1} & 0 \\ c & d \end{pmatrix}$$

and we see $g \in NeB$.

3.6 The Borel-Weil theorem for SL(n,$\mathbb{C}$)

In this section we will prove the Borel-Weil theorem for SL(n,$\mathbb{C}$). We will give all irreducible representations SL(n,$\mathbb{C}$), or to be more precise: we will give a representation for each equivalence class of irreducible representations of SL(n,$\mathbb{C}$). The proof will be a base for the proof of the Borel-Weil theorem in the general case. Because we are working with a concrete group, we will be able to show the complete structure of the represention s and decomposition of SL(n,$\mathbb{C}$).

- First note that SL(n,$\mathbb{C}$) is the complexification of $G =$ SU(n,$\mathbb{C}$). Indeed, the Lie algebra of SU(n,$\mathbb{C}$) is su(n,$\mathbb{C}$) $= \{ X \in \mathrm{M(n,}\mathbb{C}) \mid \mathrm{tr}\, X = 0,\ X + X^\dagger = 0 \}$ so su(n,$\mathbb{C}$)$^{\mathbb{C}}$ = sl(n,$\mathbb{C}$) = su(n,$\mathbb{C}$)$\oplus i$su(n,$\mathbb{C}$). We take for T the diagonal matrices in SU(n,$\mathbb{C}$). T is a Cartan subgroup of SU(n,$\mathbb{C}$). The Lie algebra $\mathfrak{t}$ of T is the collection of diagonal matrices in su(n,$\mathbb{C}$). We define $\mathfrak{h} = \mathfrak{t}^{\mathbb{C}}$.

- We have SL(n,$\mathbb{C}$) $= \{ g \in \mathrm{GL}(n, \mathbb{C}) \mid \det g = 1 \}$ and sl(n,$\mathbb{C}$) $= \{ X \in M(n, \mathbb{C}) \mid \mathrm{tr}\, X = 0 \}$. We take $H = \{ g \in \mathrm{SL(n,}\mathbb{C}) \mid g_{ij} = 0 \text{ if } i \neq j \}$ as a maximal torus (maximal abelian subgroup) in SL(n,$\mathbb{C}$). The Lie algebra of H is equal to $\mathfrak{h} = \{ X \in \mathrm{sl(n,}\mathbb{C}) \mid X_{ij} = 0 \text{ if } i \neq j \}$. We write E_{ij} for the matrix $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$. We define the linear functionals ϵ_i on $\mathfrak{h}$ by $\epsilon_i(X) = X_{ii}$. The roots of sl(n,$\mathbb{C}$) are the $\alpha_{i,j} = \epsilon_i - \epsilon_j$, $i \neq j$ with corresponding root spaces $\mathfrak{g}_{i,j} = \mathfrak{g}_{\alpha_{i,j}} = \mathbb{C}E_{ij}$. As a set of positive roots P we choose the $\alpha_{i,j}$ for which $i > j$. We have the decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h} \oplus \bigoplus_{\alpha \in P} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in P} \mathfrak{g}_{-\alpha}.$$

  We define $\mathfrak{n} = \bigoplus_{\alpha \in P} \mathfrak{g}_\alpha$, $\bar{\mathfrak{n}} = \bigoplus_{\alpha \in P} \mathfrak{g}_{-\alpha}$, $\mathfrak{b} = \mathfrak{h} \oplus n$, $\bar{\mathfrak{b}} = \mathfrak{h} \oplus \bar{\mathfrak{n}}$. We define N, B^+ and $B^- = B$ to be the Lie groups generated by $\mathfrak{n}$, $\mathfrak{b}$ and $\bar{\mathfrak{b}}$ respectively. The group N is equal to the group of upper triangular matrices with 1 on the diagonal, the group B is equal to the group of lower triangular matrices with determinant 1.

- We have determined the root-space decomposition of SL(n,$\mathbb{C}$). The next step is to calculate all the (dominant) weights of T. We know that if $X = \mathrm{diag}(X_1, X_2, \ldots, X_n)$ is an element of sl(n,$\mathbb{C}$) then $\exp X = \mathrm{diag}(e^{X_1}, e^{X_2}, \ldots, e^{X_n}) \in$ SL(n,$\mathbb{C}$). The T-lattice is

$$\Lambda = \ker \exp|_{\mathfrak{t}} = \{ X \in \mathrm{sl(n,}\mathbb{C}) \mid X_j \in 2\pi i\mathbb{Z},\ 1 \leq j \leq n \}.$$

  The weights of T are equal to

$$\hat{T} = \{ \mu_k \in i\mathfrak{t}^* \mid \mu_k(X) = \sum_j k_j X_j, \text{ for } k_j \in \mathbb{Z} \}.$$

  Note that because $\sum_i X_i = 0$ we have a certain freedom in the choice of the κ_j when defining a functional κ. We can always choose $0 \leq \sum_i k_i < n$ and this choice defines the κ_j uniquely. From the roots $\alpha \in P$ we can calculate the co-roots: because $[\mathfrak{g}_{i,j}, \mathfrak{g}_{j,i}] = \mathbb{C}(E_{ii} - E_{jj})$ we find $\alpha_{i,j}^\vee = E_{ii} - E_{jj}$. The value of a weight μ_k on a co-root $\alpha_{i,j}^\vee$ is

$$\mu_k(\alpha_{i,j}^\vee) = \sum_l k_l (\alpha_{i,j})_{ll} = k_i - k_j.$$

  The weight μ_k is dominant if it is positive on all positive co-roots, so we must have $k_i \geq k_j$ for all $i > j$.

- From the weights we get all the characters on T. If $t = \exp X = \mathrm{diag}(t_1, t_2, \ldots, t_n)$ then

$$\begin{aligned} \chi_k = \chi_{\mu_k} &: T \to \mathbb{C}^* \\ &: t \to t^{\mu_k} = e^{\mu_k(X)} = (t_1)^{k_1} \cdots (t_n)^{k_n}. \end{aligned}$$

By theorem 3.3.4 the character χ_k on T extends to a representation χ_k on B, it is given by

$$\chi_k : \begin{pmatrix} b_1 & & & \emptyset \\ \vdots & \ddots & & \\ \vdots & & \ddots & \\ * & \dots & \dots & b_n \end{pmatrix} \to b_1^{k_1} b_2^{k_2} \cdots b_n^{k_n}.$$

For example for sl(2,$\mathbb{C}$) the characters are the functions $\chi : b \to b_1^n b_2^m = b_1^{(n-m)}$. The characters corresponding to the dominant weights are the characters for which $n - m \geq 0$. Note that in the example in section 2.4 we have that exactly the anti-dominant characters give non-trivial representations of SL(2,$\mathbb{C}$). This is because we use B^+ instead of B^- in the example.

- The next step is to take a character χ_k and show that this character induces a representation of SL(n,$\mathbb{C}$) with highest weight $\lambda = \mu_k$. The induced representation $\pi = \pi_\lambda = \mathrm{ind}_B^{\mathrm{SL(n,\mathbb{C})}}(\chi_k)$ is a representation of SL(n,$\mathbb{C}$) in $\mathcal{O}(\mathrm{SL(n,\mathbb{C})}/B, L_\lambda)$ where the vector bundle $L_\lambda = \mathrm{SL(n,\mathbb{C})} \times_B \mathbb{C}$ is defined by (3.2).

  By theorem 3.2.1 we have that $\mathrm{SU(n,\mathbb{C})}/T \cong \mathrm{SL(n,\mathbb{C})}/B$. Because SU(n,$\mathbb{C}$) is compact it follows that also $\mathrm{SL(n,\mathbb{C})}/B$ is compact. From theorem 1.3.3 it follows that the space of holomorphic sections $\mathcal{O}(L_\lambda)$ is finite-dimensional. This means that the representation π is finite-dimensional.

  Now suppose we can take a highest weight vector ϕ of the representation π (such a vector exists if π is non-trivial; it is unique up to scalar multiplication if the representation π is irreducible). By definition $\pi_*(X)\phi = 0$ for all $X \in \mathfrak{g}_\alpha$, $\alpha \in P$, so $\mathfrak{n} \cdot \phi = 0$. This means that ϕ is invariant under the action of N: $N \cdot \phi = \phi$. The space of N-invariant functions is denoted by $\mathcal{O}(L_\lambda)^N$. Because ϕ is a holomorphic function on $\mathrm{SL(n,\mathbb{C})}/B$, ϕ is uniqely determined by its values on the dense subset NeB of $\mathrm{SL(n,\mathbb{C})}/B$ (NeB is dense in $\mathrm{SL(n,\mathbb{C})}/B$ by theorem 3.5.1). Every N-invariant function is therefore already characterized by its value on $e \in N$. We can define the linear mapping $\mathrm{ev}_e : \mathcal{O}(L_\lambda) \to (L_\lambda)_e$ by sending ϕ to $\phi(e)$. Because every N-invariant function is determined by its value on e, the restriction

  $$\mathrm{ev}_e : \mathcal{O}(L_\lambda)^N \to (L_\lambda)_e \cong \mathbb{C}$$

  is injective. Because ev_e is injective on $\mathcal{O}(L_\lambda)^N$, we must have $\dim \mathcal{O}(L_\lambda)^N \leq 1$. From theorem 3.4.1 it follows that the representation π is irreducible.

So far we have not used anything specific of the group $G^{\mathbb{C}} = \mathrm{SL(n,\mathbb{C})}$. All statements can easily be generalized to a general compact, connected, reductive group G. This will be done in the section 3.7.

For every weight λ we have constructed an irreducible or trivial representation π_λ of G in the space of holomorphic sections, or in the space $\mathcal{O}(\mathrm{SL(n,\mathbb{C})}, \mathbb{C}, \chi_k)$. Now we have to show that the representation π_λ is non-trivial if and only if λ is a dominant weight. We also have to prove that the left regular representation π is a representation with highest weight λ.

For SL(n,$\mathbb{C}$) we will first prove that the representation π has λ as its highest weight and that this weight must be dominant for the representation to be non-trivial. After that we will prove that for every dominant weight the representation is non-trivial by constructing a highest weight vector in the space $\mathcal{O}(\mathrm{SL(n,\mathbb{C})}, \mathbb{C}, \chi_k)$. In the general case this is not possible and we have to use another proof. An outline of this proof is given in the next section, for now we concentrate on SL(n,$\mathbb{C}$).

- We assume $\mathcal{O}(G^{\mathbb{C}}/B, L_\lambda)$ is non-zero, so we can take a section $s \in \mathcal{O}(G^{\mathbb{C}}/B, L_\lambda)^N$. To this section there is a corresponding function $f \in \mathcal{O}(G^{\mathbb{C}}, \mathbb{C}, \chi_\lambda)$ which is a highest weight vector of the representation π_λ. The function f transforms under the action of G according to a weight μ. The function f has several transformation properties:

– Because $f \in \mathcal{O}(G^{\mathbb{C}}/B, \mathbb{C}, \chi_\lambda)$ we have for every $x \in G^{\mathbb{C}}$, $h \in B$

$$f(xh) = \chi_\lambda(h)^{-1} f(x) = h^{-\lambda} f(x).$$

– Because f is a highest weight vector for the weight μ for the left regular representation of $G^{\mathbb{C}}$, f transforms as

$$f(hx) = [\pi_\lambda(h^{-1})f](x) = h^{-\mu} f(x)$$

for a weight μ and $x \in G^{\mathbb{C}}$, $h \in H$.

– For $n \in N$ we have $f(nx) = n^{-\mu} f(x) = f(x)$ (because f is in $\mathcal{O}(G^{\mathbb{C}}/B, L_\lambda)^N$).

From the transformation properties it follows that because $f \neq 0$ we must have $f(e) \neq 0$. We also have for every $h \in B$ that

$$f(e) = f(hh^{-1}) = h^{-\mu} h^{\lambda} f(e) = h^{\lambda-\mu} f(e).$$

This implies that $\mu = \lambda$, so the representation π_λ is indeed a representation with highest weight λ.

We must now prove that $\lambda = \mu$ is dominant. For this we notice that for every weight α we have that $\mathfrak{g}(\alpha) = \mathfrak{g}_\alpha \oplus \mathbb{C}\alpha^\vee \oplus \mathfrak{g}_{-\alpha}$ is isomorphic to sl(2,$\mathbb{C}$). We can select an sl(2,$\mathbb{C}$)-triple X_α, Y_α, H_α with $X_\alpha \in \mathfrak{g}_\alpha$, $Y_\alpha \in \mathfrak{g}_{-\alpha}$ and $H_\alpha = \alpha^\vee \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$. If (π_*, V) is the representation of $\mathfrak{g}^{\mathbb{C}}$ corresponding to π_λ (defined by $\pi_* = T_e\pi_\lambda$) then $\mathfrak{h}$ acts on $V^{\mathfrak{n}}$ through the weight λ. In particular we have for every $X \in \mathfrak{h}$ that $\pi_*(H) - \lambda(H)I = 0$ on $V^{\mathfrak{n}}$.

We can now consider $\pi_*|_{\mathfrak{g}_\alpha}$ for a positive root α. Because $\mathfrak{g}_\alpha \subset \mathfrak{n}$, we have that $V^{\mathfrak{n}} \subset V^{\mathfrak{g}_\alpha} = V^{\mathfrak{n}(\alpha)}$. This means that every vector in $V^{\mathfrak{n}}$ is a highest weight vector for $\mathfrak{g}(\alpha)$. Now $\lambda(H_\alpha)$ is positive, because λ is a highest weight (and therefore dominant) for the sl(2,$\mathbb{C}$)-representation of $\mathfrak{g}(\alpha)$. So for every positive weight α we have that $\lambda(H_\alpha) \geq 0$, in other words: λ is dominant.

We will now look some closer at the weights and characters. By considering only the fundamental weights it will be more easy to construct a function f which is a highest weight vector for the representation π_λ.

- Note that if λ, λ_1 and λ_2 are all weights and $\lambda = \lambda_1 + \lambda_2$ then we have $\xi = \xi_1\xi_2$ if ξ, ξ_1 and ξ_2 are the characters corresponding to λ, λ_1 and λ_2. We also have that if $f_1 \in \mathcal{O}(G^{\mathbb{C}}, \mathbb{C}, \xi_1)$ and $f_2 \in \mathcal{O}(G^{\mathbb{C}}, \mathbb{C}, \xi_2)$ then $f = f_1 f_2 \in \mathcal{O}(G^{\mathbb{C}}, \mathbb{C}, \xi)$ because for $x \in G^{\mathbb{C}}$ and $h \in B$ we have

$$\begin{aligned} f(xh) &= f_1(xh) f_2(xh) = \xi_1(h)^{-1} f_1(x) \xi_2(h)^{-1} f_2(x) \\ &= (\xi_1(h)\xi_2(h))^{-1} f_1(x) f_2(x) \\ &= \xi(h)^{-1} f(x). \end{aligned}$$

This means that we only have to construct a holomorphic function for every fundamental weight because for an arbitrary weight the function is a product of functions for the fundamental weight.

The fundamental weights are the weights $\lambda_j = \sum_{k=1}^{j} \epsilon_k$ for $1 \leq j \leq n-1$. To these fundamental weights correspond the fundamental characters:

$$\chi_j : B \to \mathbb{C}^* : \begin{pmatrix} t_1 & & & \\ \vdots & \ddots & & \emptyset \\ \vdots & & \ddots & \\ * & \dots & \dots & t_n \end{pmatrix} \to \prod_{k=1}^{j} t_k.$$

We define for $1 \leq j \leq n$ the functions

$$A_j(x) = \det \begin{pmatrix} x_{jj} & \dots & x_{jn} \\ \vdots & & \vdots \\ x_{nj} & \dots & x_{nn} \end{pmatrix}.$$

The function A_j is a polynomial function, so $A_j \in \mathcal{O}(\mathrm{SL(n,\mathbb{C})})$. We also have for $x \in \mathrm{SL(n,\mathbb{C})}$ and $h \in B$

$$\begin{aligned} A_j(xh) &= \det \begin{pmatrix} x_{jj} & \dots & x_{jn} \\ \vdots & & \vdots \\ x_{nj} & \dots & x_{nn} \end{pmatrix} \begin{pmatrix} h_{jj} & & \emptyset \\ \vdots & \ddots & \\ * & \dots & h_{nn} \end{pmatrix} \\ &= \det \begin{pmatrix} x_{jj} & \dots & x_{jn} \\ \vdots & & \vdots \\ x_{nj} & \dots & x_{nn} \end{pmatrix} \det \begin{pmatrix} h_{jj} & & \emptyset \\ \vdots & \ddots & \\ * & \dots & h_{nn} \end{pmatrix} \\ &= A_j(x)A_j(h) = A_j(x) \prod_{k=j}^{n} h_{kk} \\ &= (\prod_{k=1}^{j-1} h_{kk})^{-1} A_j(x) = \chi_{j-1}(h)^{-1} A_j(x) \end{aligned}$$

and so $A_j \in \mathcal{O}(\mathrm{SL(n,\mathbb{C})}, \mathbb{C}, \chi_{j-1})$.

- It is not difficult to see that the functions A_j we have defined here are invariant under the action of N. This means that the A_j are highest weight vectors. We also know that there is only one highest weight vector (up to scalar multiplication). Because we have already proved that the A_j are functions in $\mathcal{O}(\mathrm{SL(n,\mathbb{C})}, \mathbb{C}, \chi_j)^N$ it follows from the previous points that the A_j are highest weight vectors for the weight λ_j.

- For every fundamental weight λ_j we now have a function A_{j+1} in $\mathcal{O}(L_{\lambda_j})$. For a dominant weight $\lambda = \sum k_j \lambda_j$ we have that

$$f = \prod_{j=1}^{n-1} A_{j+1}^{k_j}$$

is a function in $\mathcal{O}(L_\lambda)$. This means that f is a highest weight vector with highest weight λ.

In this section we have proven the following theorem:

Theorem 3.6.1 (Borel-Weil for SL(n,ℂ))**.** *Every irreducible representation of SL(n,ℂ) is isomorphic to the left regular representation of SL(n,ℂ) on the space* $\mathcal{O}(L_\lambda) \cong \mathcal{O}(G^{\mathbb{C}}, \mathbb{C}, \xi_\lambda)$ *for a unique dominant weight* λ*.*

In particular, every representation of SL(n,ℂ) is uniquely characterized by a $\mathbf{k} \in \{ \mathbf{k} \in \mathbb{Z}^n_{\geq 0} \mid k_j \geq k_{j+1} \}$.

Example 6. In our proof it was already clear that the functions A_j are highest weight vectors for the weight λ. We can also direcly verify that the A_j are highest weight vectors.

To show this we need to look at the representation π_*, which is defined by $\pi_* = T_e\pi$. We prove that A_j is an element of the weight space

$$V_{\lambda_j} = \{ f \in \mathcal{O}(\mathrm{SL(n,\mathbb{C})}, \mathbb{C}, \chi_j) \mid \pi_*(X)f = \lambda_j(X)f, \ \forall X \in \mathfrak{h} \}.$$

We have for $z \in \mathrm{SL}(,\mathbb{C})$ and $X \in H$:

$$
\begin{aligned}
[\pi_*(X)A_j](z) &= \frac{d}{dt}\Big|_{t=0} \pi(\exp tX)A_j(z) = \frac{d}{dt}\Big|_{t=0} A_j(\exp(-tX)z) \\
&= \sum_{k,l} \frac{\partial A_j}{\partial z_{kl}} (-Xz)_{kl} \\
&= \sum_{k,l\geq j} (-1)^{k+l} \det \begin{pmatrix} z_{j,j} & \cdots & z_{j,k-1} & z_{j,k+1} & \cdots & z_{j,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ z_{k-1,j} & \cdots & z_{k-1,k-1} & z_{k-1,k+1} & \cdots & z_{k-1,n} \\ z_{k+1,j} & \cdots & z_{k+1,k-1} & z_{k+1,k+1} & \cdots & z_{k+1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ z_{n,j} & \cdots & z_{n,k-1} & z_{n,k+1} & \cdots & z_{n,n} \end{pmatrix} (-X_{kk}z_{kl}) \\
&= -\sum_{k\geq j} X_{kk} \sum_{l\geq j} (-1)^{k+l} z_{kl} \det \begin{pmatrix} z_{j,j} & \cdots & z_{j,k-1} & z_{j,k+1} & \cdots & z_{j,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ z_{k-1,j} & \cdots & z_{k-1,k-1} & z_{k-1,k+1} & \cdots & z_{k-1,n} \\ z_{k+1,j} & \cdots & z_{k+1,k-1} & z_{k+1,k+1} & \cdots & z_{k+1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ z_{n,j} & \cdots & z_{n,k-1} & z_{n,k+1} & \cdots & z_{n,n} \end{pmatrix} \\
&= -\sum_{k\geq j} X_{kk} \det \begin{pmatrix} z_{j,j} & \cdots & z_{j,n} \\ \vdots & \ddots & \vdots \\ z_{n,j} & \cdots & z_{n,n} \end{pmatrix} \\
&= -\sum_{k\geq j} X_{kk} A_j(z) = (\sum_{k<j} X_{kk}) A_j(z) = \lambda_j(X) A_j(z).
\end{aligned}
$$

This means that $A_j \in V_{\lambda_j}$, so A_j is a highest weight vector for the weight λ_j.

Example 7. As a short example, we will construct a highest weight vector of the representation of SL(3,$\mathbb{C}$) in the space $\mathcal{O}(\mathrm{SL}(3,\mathbb{C}), \mathbb{C}, \xi)$ where ξ is the representation corresponding to the dominant weight μ defined by

$$
\mu : \begin{pmatrix} X_1 & 0 & 0 \\ 0 & X_2 & 0 \\ 0 & 0 & X_3 \end{pmatrix} \to 3X_1 + 2X_2.
$$

For SL(3,$\mathbb{C}$) there are two fundamental weights λ_1 and λ_2 defined by $\lambda_j(X) = \sum_{k=1}^{j} X_k$. We have $\mu = \lambda_1 + 2\lambda_2$. The characters are

$$
\begin{aligned}
\xi_1 : t = \begin{pmatrix} t_1 & 0 & 0 \\ * & t_2 & 0 \\ * & * & t_3 \end{pmatrix} &\to t_1, \\
\xi_2 : t &\to t_1 t_2, \\
\xi = \xi_1 \xi_2^2 : t &\to t_1^3 t_2^2.
\end{aligned}
$$

As before we can define the functions A_j as subdeterminants of a matrix.

$$
\begin{aligned}
A_1(x) &= \det x = 1, \\
A_2(x) &= \det \left(\begin{smallmatrix} x_{22} & x_{23} \\ x_{32} & x_{33} \end{smallmatrix} \right) = x_{22}x_{33} - x_{32}x_{23}, \\
A_3(x) &= \det \big(x_{33} \big) = x_{33}.
\end{aligned}
$$

For a matrix $\left(\begin{smallmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{smallmatrix}\right) \in B$ we have $A_1(xh) = A_1(x) = 1$, $A_2(xh) = cfA_2(x) = a^{-1}A_2(x)$ and $A_3(xh) = fA_3(x) = (ac)^{-1}A_3(x)$. If we define

$$g(x) = (A_2A_3^2)(x) = x_{33}^2(x_{22}x_{33} - x_{32}x_{23})$$

then $g(xh) = \xi(h)^{-1}g(x)$. It is also not difficult to show that g is a highest weight vector with weight μ.

3.7 The Borel-Weil theorem in the general case

In this section we present an outline of the proof of the Borel-Weil theorem in a more general case such as it is given in [4]. There are however many variations on the proof and the statement of the theorem.

Theorem 3.7.1 (Borel-Weil)**.** *Let G be a compact connected Lie group, T a maximal torus in G. For every weight λ define the subgroup B and the holomorphic line bundle L_λ as in (3.2). Let π_λ be the representation of G (or $G^{\mathbb{C}}$) on $\mathcal{O}(G^{\mathbb{C}}/B, L_\lambda)$ defined by left translation.*

Then the representation π_λ is non-trivial ($\mathcal{O}(G^{\mathbb{C}}/B, L_\lambda) \neq 0$) if and only if λ is a dominant weight. If λ is dominant, then π_λ defines an irreducible representation of G with highest weight λ.

Outline of proof of the Borel-Weil theorem. In the proof we will make use of the highest weight theorem (theorem 3.3.3). First take $\mathfrak{g}$ to be the Lie algebra of G. Let $G^{\mathbb{C}}$ be the complexification of G and $\mathfrak{g}^{\mathbb{C}}$ the complexification of $\mathfrak{g}$. Take $\mathfrak{t} = \mathrm{Lie}(T)$ and $\mathfrak{h} = \mathfrak{t}^{\mathbb{C}}$. Fix a choice of positive roots P. Define $\mathfrak{n} = \oplus_{\alpha>0}\mathfrak{g}_\alpha$ and $\bar{\mathfrak{b}} = \mathfrak{h} \oplus_{\alpha>0} \mathfrak{g}_{-\alpha}$. Take N and $B = B^-$ to be the subgroups of $G^{\mathbb{C}}$ with Lie algebras $\mathfrak{n}$ and $\bar{\mathfrak{b}}$ respectively. Now we have introduced the notation that will be used, we can begin with the proof.

- If G is simply connected we can indentify the representations of G and the holomorphic representations of $G^{\mathbb{C}}$ by using theorem 3.2.2. If G is not simply connected the situation is a bit more complicated and it requires more work to show that the representations of G and $G^{\mathbb{C}}$ are equivalent. We refer to [4, Lemma 5.7] for the details.

- First note that for every weight λ we can construct the line bundle L_λ defined by (3.2). In the space of holomorphic sections of L_λ we have a representation $\pi = \pi_\lambda$ of $G^{\mathbb{C}}$ defined by formula (2.1). This representation is irreducible, the proof is analogous to the proof for the case of SL(n,$\mathbb{C}$). We assume π is non-trivial, so we can take a non-zero section $s \in \mathcal{O}(L_\lambda)^N$. To this section corresponds a function $f \in \mathcal{O}(G, \mathbb{C}, \xi_\lambda)$. This function is invariant under the action of N and is equivariant under the action of B. So this function is uniquely determined on the set NB by its value in e. By theorem 3.5.1 the function is now uniquely determined on the entire $G^{\mathbb{C}}$ by its value on e. This means that the mapping $\mathrm{ev} : \mathcal{O}(L_\lambda)^N \to \mathbb{C} : s \to s(e)$ is injective, so $\dim \mathcal{O}(L_\lambda)^N \leq 1$. By theorem 3.4.1 the representation π is irreducible.

- By the highest-weight theorem there is an irreducible representation (ρ, V) of G with highest weight λ for every dominant weight λ. This representation can be chosen unitary for an inner product on V because G is compact. From now on we assume such an inner product $\langle \cdot, \cdot \rangle$ to be chosen. We can extend this representation to a holomorphic representation ρ of $G^{\mathbb{C}}$ in V using theorem 3.2.2.

  Now suppose v_λ is a dominant weight for ρ. We can define the mapping

  $$T : V \to \mathcal{O}(G) : v \to \psi_v \tag{3.3}$$

  where ψ_v is defined by

  $$\psi_v(x) = \langle \rho(x)^{-1}v, v_\lambda \rangle.$$

  One can prove the following properties of the map T:

- T is linear and injective.
- Every function ψ_v satisfies $\psi_v(xh) = \xi_\lambda(h)^{-1}\psi_v(x)$ for $x \in G$ and $h \in B$. This means that T is in fact a map $V \to \mathcal{O}(L_\lambda) = \mathcal{O}(G, \mathbb{C}, \xi_\lambda)$.
- For $x \in G$ we have $\pi(x)\psi_v = \psi_{\rho(x)v}$ so T is a mapping intertwining the actions of ρ and π.

- Because T intertwines the actions, the image of T in $\mathcal{O}(L_\lambda)$ is an irreducible subspace of $\mathcal{O}(L_\lambda)$ which is invariant under the action of G. Because the space $\mathcal{O}(L_\lambda)$ is irreducible we must have $\operatorname{Im} T = \{0\}$ or $\operatorname{Im} T = \mathcal{O}(L_\lambda)$. It is easy to see that $\operatorname{Im} T \neq \{0\}$ (just look at $T(v_\lambda) = \psi_{v_\lambda}$), and therefore $\operatorname{Im} T = \mathcal{O}(L_\lambda)$. The mapping T is thus a bijective mapping from V onto $\mathcal{O}(L_\lambda)$ and it intertwines the actions of ρ and π. This means that (ρ, V) and $(\pi, \mathcal{O}(L_\lambda))$ are equivalent representations.
- We have proven that (ρ, V) and $(\pi, \mathcal{O}(L_\lambda))$ are equivalent, so π is an irreducible representation with highest weight λ. This completes the proof.

$\square$

Appendix A

Basic definitions

A.1 Complex manifolds

In this section we give some basic definitions of complex manifolds. Most of these definitions are taken from [3]. The easiest way to think of a complex manifold is as a smooth (C^∞) manifold which is locally homeomorphic to $\mathbb{C}^n$ instead of $\mathbb{R}^n$ and for which all the coordinate transformations are analytic (holomorphic) functions. With this in mind we can define a complex manifold.

Definition A.1.1. A smooth manifold X of dimension $2n$ is called a complex analytic manifold of complex dimension n if there is a family $\mathcal{F}$ of homeomophisms κ, called complex analytic coordinate systems, op open sets $X_\kappa \subset X$ onto open sets $\tilde{X}_\kappa \subset \mathbb{C}^n$ such that

(i) If $\kappa, \kappa' \in \mathcal{F}$ then the mapping

$$\kappa' \kappa^{-1} : \kappa(X_\kappa \cap X_{\kappa'}) \to \kappa'(X_\kappa \cap X_{\kappa'})$$

is analytic.

(ii) The union of all X_κ for $\kappa \in \mathcal{F}$ is equal to X.

We can now define what holomorphic mappings and function are on a complex manifold just as for a C^r manifold we can define C^r functions and mappings.

Definition A.1.2. Let X_1 and X_2 be complex analytic manifolds. Then a mapping $f : X_1 \to X_2$ is called analytic if $\kappa_2 \circ f \circ \kappa_1^{-1}$ is analytic (where is is defined) for all coordinate systems κ_1 in X_1 and κ_2 in X_2.

An analytic function on a complex analytic manifold X is just an analytic mapping from X to $\mathbb{C}$ (which is cleary a complex analytic manifold). For complex manifolds we can define other concepts such as submanifolds in a way similar to the case of real manifolds.

A.2 Complex Lie groups and Lie algebras

Definition A.2.1. A (complex) Lie group is a smooth (complex) manifold G that is at the same time a group G for which the group operations $(x, y) \to xy$ and $x \to x^{-1}$ are smooth (holomorphic).

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