Cauchy-Kowalevski and Cartan-Kähler

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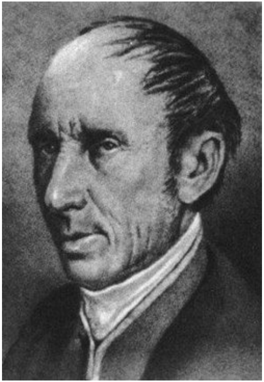
Outline

- Cauchy-Kowalevski theorem
- Partial differential equations and exterior differential systems
- Cartan-Kähler theorem

Augustin Louis Cauchy

- Born August 21, 1789, died May 23, 1857
- Analysis, complex analysis, permutation groups.



A first order quasi-linear PDE

Theorem (Cauchy)

The analytic partial differential equation

$$\frac{\partial u}{\partial t}(x,t) = \sum_j f_j(x,t,u)\frac{\partial u}{\partial x^j}(x,t) + g(x,t,u), \quad u(x,0) = \psi(x).$$

has a unique solution $u(x,t)$.

- Note that the variable t (often associated with time) has a distinct role.
- The theorem is also true for vector-valued functions u.

Higher order PDE's

Consider the wave equation $\frac{\partial^2 u}{\partial x \partial t} = 0$. We can rewrite this equation as a first order system of equations. We let $v = \frac{\partial u}{\partial x}$.

$$\frac{\partial u}{\partial x} = v,$$
$$\frac{\partial v}{\partial t} = 0.$$

General strategy: replace the derivatives of the unknown functions u by new unknown functions. Add equations to the system that describe the relations. This can be used to generalize the previous theorem to higher order and non-linear PDE's. The result is the Cauchy-Kowalevski theorem.

Sofie Kowalevski

Sonya Kovalevskaya.

Sofie Kowalevski

- Born in 1850, Moscow, Russia. Started mathematics at a young age.
- In 1868 married Vladimir Kovalevski to be able to study mathematics and in 1869 moved to Heidelberg
- In 1871 she moved to Berlin to study with Weierstrass. Published 3 papers but could not get a position mainly because of her gender. Wrote novels for several years and finally obtained a position in Stockholm.
- First woman to receive a doctorate in mathematics and the first woman to obtain a permanent position on a university faculty in mathematics.

The MathSciNet list

Kovalevskaya, Sof'ya Vasil'evna ; Kovalevskaya, S. V. ; Kovalevskaya, Sofya ; Kovalevskaya, Sof'ya ; Kovalevskaja, Sophia Vasilievna ; Kovalevskaja, S. V. ; Kovalevskaja, Sof'ja ; Kovalevsky, Sofya ; Kovalevskaya, Sofia ; Kovalevskaja, Sonya ; Kovalevskaja, Sofya ; Kovalevsky, Sophie ; Kovalevsky, Sonja ; Kowalevski, Sophie ; Kovalevskaja, Sof'ya ; Kovalevskaja, Sof'ja ; Kovalevskaya, Sonia ; Kovalevskaia, S. ; Kovalevskaya, Sonya ; Kovalevskaia, Sofia ; Kovalevskaja, Sonja ; Kovalevsky, Sophia ; Kovalevskaia, Sofia Vasilevna ; Kovalevskoja, S. ; Kovalevsky, S. ; Kowalewskaja, Sofja ; Kowalewskaja, Sofia ; Kovalevskaya, Sophia

Cauchy-Kowalevski theorem

Theorem (Cauchy-Kowalevski theorem)

The analytic partial differential equation

$$\frac{\partial^n u}{\partial t^n} = F(x, t, u, \frac{\partial u}{\partial x}, \ldots, \frac{\partial^\beta}{\partial x^\beta}\frac{\partial^l u}{\partial t^l}).$$

has a unique analytic solution with boundary conditions

$$u(x,0) = \phi(x),\ \frac{\partial u}{\partial t}(x,0) = \phi_1(x), \ldots,\ \frac{\partial^{n-1} u}{\partial t^{n-1}}(x,0) = \phi_{n-1}(x).$$

Here $x \in \mathbb{R}^n$ and $u(x,t)$ is a function on $\mathbb{R}^n \times \mathbb{R}$. We can allow u to be vector valued, but for simplicity we will omit the indices.

Proof (by Kowalevski and Cauchy)

- First convert the problem to the case of a first order quasi-linear equation.
- By considering powers of x and t one quickly sees that there is a unique formal power series solution.
- We are left with proving convergence of the power series solution. This is a straightforward (but not very interesting) application of Cauchy's majorant method. The estimates on the boundary conditions and the function F translate to estimates on the formal power series solution.

Modern proof (Ovsjannikov)

- A modern proof would consider the equation

$$\frac{\partial u}{\partial t} = F(t, u), \quad u(0) = v$$

  where F is a differential operator and $u(t)$ is valued in a suitable space of functions. The space is provided with a suitable norm (or family of norms) to make it into a Banach space.

- Next we apply Picard iteration as we would for an ordinary differential equation, i.e. we consider

$$u_{n+1}(t) = v + \int_0^t F(u_n(\tau), \tau) d\tau$$

  and prove convergence.

Generalizations

- In general PDE's are difficult. The Cauchy-Kowalevski theorem is one of the very few general theorems. One of the reasons for this is that the type of equation (for example elliptic or hyperbolic) determines the type of solutions and the type of boundary conditions one can set.
- A major disadvantage of the Cauchy-Kowalevski theorem is that it is only true in the analytic setting. There are many counterexamples in the C^∞ setting!

Skip heat equation example

- We give one counterexample that was already given by Kowalevski: the heat equation.

The heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

- We can apply the Cauchy-Kowalevski theorem if we set initial values on the x variable, i.e. we can prescribe $u(0, t) = \phi(t)$.
- But this is not very natural, we would like to prescribe the temperature at a fixed moment in time and see how the temperature evolves!
- So suppose we want to have $u(x, 0) = \psi(x)$ and solve the heat equation. How do we do this?

Power series for the heat equation

- By writing a formal series as

$$u(x,t) = \sum c_{kl} x^k t^l$$

and substituting this into the heat equation we can deduce the relations

$$c_{kl} = \frac{(k+2l)!}{k!l!} c_{k+2l,0}$$

Note that the coefficients $c_{k+2l,0}$ are determined by the initial value $\psi(x)$!

- So any analytic solution will have to satisfy these relations.

Initial conditions

- We start with a innocently looking initial function $\psi(x) = 1/(1-x)$. Near 0 this is an analytic function in x. The series expansion of $\phi(x)$ is $1 + x + x^2 + \ldots$.
- Any analytic solution u of the heat equation with $u(x,0) = \psi(x)$ should therefore satisfy

$$c_{k0} = 1$$
$$c_{kl} = \frac{(k+2l)!}{k!l!}.$$

  This determines a unique formal power series.
- Suppose we analyze converge of this power series at a point $(x,t) = (0,\epsilon)$. We then have

$$u(0,\epsilon) = \sum_l \frac{(2l)!}{l!}\epsilon^l.$$

  This series diverges for all $\epsilon > 0$!

Exterior differential systems

Definition

We denote the k-forms on a manifold M by $\Omega^k(M)$. The k-forms form a vector space.

On differential forms we have two very important operations:

- The wedge product.

$$(dx + dy) \wedge dy = dx \wedge dy + dy \wedge dy = dx \wedge dy$$

- The exterior differentiation operation.

$$d(x^2 + y^2) = 2xdx + 2ydy$$
$$d(xdy) = dx \wedge dy$$

Exterior differential systems

Definition

The k-forms together with the wedge product define an algebra, which we denote by $\Omega^*(M)$.

An ideal in the algebra is a subset that is closed under addition and taking wedge products with arbitrary k-forms. An ideal is differentially closed if the ideal is closed under the d-operator.

Definition

An exterior differential system on a manifold M is a differentially closed ideal in $\Omega^*(M)$.

Integral manifolds

Definition

Let $\mathcal{I}$ be an exterior differential system on the manifold M spanned by forms $\theta^1, \ldots, \theta^a$. An integral manifold U for the system is a submanifold such that if $\phi : U \to M$ is an embedding we have

$$\phi^*(\theta^a) = 0.$$

Example: integral curves

Example

Let $M = \mathbb{R}^2$ and consider the vector field $(-y, x)$. We define the 1-form $\theta = xdx + ydy$. The integral manifolds for θ are 1-dimensional and are given by the solution curves of the vector field.

Indeed: let $\phi : t \mapsto (cos(t), sin(t))$. Then we have

$$\begin{aligned}\phi^*(\theta) &= \phi^*(xdx + ydy) \\ &= \cos(t)d(\cos(t)) + \sin(t)d(\sin(t)) \\ &= -\cos(t)\sin(t)dt + \sin(t)\cos(t)dt = 0.\end{aligned}$$

Independence conditions

- An independence condition for an exterior differential system is a n-form $\omega^1 \wedge \omega^2 \wedge \ldots \wedge \omega^n$. We require that on integral manifolds this k-form restricts to a non-zero volume form.
- For example we can take $\omega^1 = dx$ as an independence condition in the previous example. The pullback is

$$\phi^*(dx) = -\sin(t)dt.$$

  This is non-zero when $t \notin \mathbb{Z}\pi$.

The wave equation

$$\Delta u = \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$$

Solutions are just functions $u(x, y)$. Consider the manifold M with coordinates $x, y, u, p = u_x, q = u_y, r = u_{xx}, s = u_{xy}, t = u_{yy}$. Note that any solution (in fact any function) defines a 2-dimensional submanifold of M by

$$U : \mathbb{R}^2 \to M : (x, y) \mapsto (x, y, u(x, y), \frac{\partial u}{\partial x}(x, y), \ldots, \frac{\partial^2 u}{\partial y^2}).$$

But what submanifolds of M are solutions of the wave equation?

On M we define three 1-forms (called the contact forms)

$$\theta^1 = du - pdx - qdy,$$
$$\theta^2 = dp - rdx - sdy,$$
$$\theta^3 = dq - sdx - tdy.$$

First property: the submanifolds U defined on the slide before satisfy the condition that

$$U^* \theta^j = 0.$$

The converse is also true: any submanifold U for which the pullbacks of the forms θ^j is zero can locally be written as the graph of a function.

Second property: any solution of the wave equation is mapped to the submanifold $\tilde{M}$ of M defined by $r - t = 0$. These two properties characterize the solutions of the wave equation.

Theorem

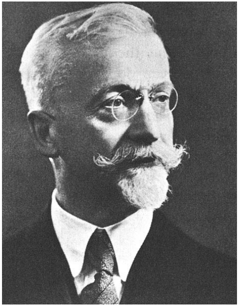
The solutions of the wave equation are (locally) in one-to-one correspondence with the integral manifolds of the exterior differential system $I = (\theta^1, \theta^2, \theta^3)$ *on* $\tilde{M}$ *with independence condition* $J = dx \wedge dy \neq 0$.

PDE's

What is the relation between PDE's and exterior differential system?

	PDE	exterior differential system
framework	local coordinates	geometric
system	system of PDE's	exterior differential ideal
solutions	functions	integral manifolds
differentiation	partial derivatives	d operator

Local coordinates are messy and often obscure the geometry of the objects we are working with.



Elie Joseph Cartan

- Born 9 April 1869 in Dolomieu, France. Died: 6 May 1951 in Paris, France
- Continuous (Lie) groups, Lie algebras, PDE's (integrable systems, prolongation), Riemannian geometry, symmetric spaces, relativity and spinors.
- Developed the theory of exterior differential forms, the Cartan-Kähler theorem and the method of equivalence.

Erich Kähler

Erich Kähler

- Born 16 January 1906, died 31 May 2000
- The Cartan-Kähler theorem on singular solutions of non-linear analytic differential systems, Kähler metric on complex manifolds, Kähler differentials.

Basic problem

We are given a linear exterior differential system I generated by set of differential 1-forms $\theta^1, \ldots, \theta^a$ and an independence condition $J = \omega^1 \wedge \omega^2 \wedge \ldots \wedge \omega^n \neq 0$. We are looking for integral surfaces of I.

The structure equations are

$$d\theta^a \equiv \pi_i^a \wedge \omega^i + T_{ij}^a \omega^i \wedge \omega^j \mod I.$$

Or without indices

$$d\theta \equiv \pi \wedge \omega + T\omega \wedge \omega \mod I.$$

Or in matrix form

$$d \begin{pmatrix} \theta^1 \\ \theta^2 \\ \vdots \end{pmatrix} \equiv \begin{pmatrix} \pi_1^1 & \pi_2^1 & \ldots \\ \pi_1^2 & \pi_2^2 & \ldots \\ \vdots & \vdots & \ddots \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \\ \vdots \end{pmatrix} + \ldots$$

Torsion

$$d\theta^a = \pi^a_i \wedge \omega^i + T^a_{ij}\omega^i \wedge \omega^j \mod I.$$

The terms T^a_{ij} are called torsion. We can sometimes change the torsion by replacing the terms π^a_i with new terms. The part that cannot be changed is called intrinsic torsion.
At the points where there is intrinsic torsion, there are no integral manifolds. The reason is that the terms $\omega^i \wedge \omega^j$ will never become zero under the pullback due to the independence condition.

Cartan characters

Definition

Let (I, J) be an exterior differential system without torsion so we can write the structure equations as

$$d\theta^a \equiv \pi_i^a \wedge \omega^i \mod I.$$

The number of independent forms π_k^a in column k is the Cartan character s_k. This number depends on the basis chosen, so we agree to choose a generic basis. This is a basis θ^a for I such that the s_k are maximal.

Cartan's test

The Cartan characters determine some algebraic numbers. Using these numbers we can test wether the system is in involution or not. This is called Cartan's test.
If a system is in involution there are no hidden integrability conditions.

The Cartan-Kähler theorem

Theorem (Cartan-Kähler)

Let (I, J) be a linear exterior differential system without torsion and Cartan characters s_k. Assume Cartan's test is satisfied and the system is in involution.
Then the system has integral manifolds and the general solution depends on s_1 functions of one variable, s_2 functions of 2 variables, $\ldots$, s_n functions of n variables.

The theorem does not give an explicit method for finding the integral manifolds. From the proof it is clear that one can find these integral manifolds by a repeated application of the Cauchy-Kowalevski theorem.

Examples

We will give two examples of the Cartan-Kähler theorem

- The wave equation
- Surfaces in $\mathbb{R}^3$

The wave equation again

Remember that solutions of the wave equation corresponded to integral manifolds of the exterior differential system I spanned by

$$\begin{aligned} \theta^1 &= du - pdx - qdy, \\ \theta^2 &= dp - rdx - sdy, \\ \theta^3 &= dq - sdx - rdy. \end{aligned}$$

with independence condition $dx \wedge dy \neq 0$. Let us calculate the structure equations ...

$$\begin{aligned} d\theta^1 &= -dp \wedge dx - dq \wedge dy \\ &\equiv (\theta^2 + rdx + sdy) \wedge dx + (\theta^3 + sdx + rdy) \\ &\equiv sdy \wedge dx + sdx \wedge dy = 0. \end{aligned}$$

In a similar way

$$d \begin{pmatrix} \theta^2 \\ \theta^3 \end{pmatrix} \equiv \begin{pmatrix} dr & ds \\ ds & dr \end{pmatrix} \wedge \begin{pmatrix} dx \\ dy \end{pmatrix}.$$

$$d \begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 \\ dr & ds \\ ds & dr \end{pmatrix} \wedge \begin{pmatrix} dx \\ dy \end{pmatrix}$$

- There is no torsion!
- The Cartan characters are $s_1 = 2, s_2 = 0$. The system is in involution.
- By the Cartan-Kähler theorem the general solution depends on two functions of 1 variable.
- For the wave equation the "dimension count" of two functions of one variable turns out to be correct.

$$u(x, y) = f(x + y) + g(x - y)$$

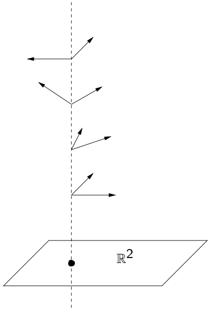
$$\frac{\partial^2 u}{\partial x \partial y} = 0$$

Surfaces in $\mathbb{R}^3$

We will look at surfaces in $\mathbb{R}^3$ and impose some conditions on the curvature of these surfaces. But first we have to formulate our problem in terms of an exterior differential system.

The orthonormal frame bundle

We let $M = \mathbb{R}^3$ with the usual metric. We can define the orthonormal frame bundle FM. A point in the frame bundle consists of a pair (x, u) where $x \in M$ and u is an isomorphism $T_x M \to \mathbb{R}^3$ that represents an orthonormal frame in $T_x M$. (A frame is a basis for the tangent space).
The bundle $FM \to M$ is a smooth fibre bundle.



On the frame bundle we can choose a canonical 1-form ω by defining

$$\omega_{x,u}(X) = u(T_{x,E}\pi(X))$$

$$\begin{array}{c} FM \\ \downarrow \pi \\ M \end{array}$$

Note the 1-form is $\mathbb{R}^3$ valued. For convenience we will choose a basis and write ω as $\omega^1, \omega^2, \omega^3$.

Theorem (Theorem of Riemannian geometry)

There are unique 1-forms ω^i_j on the frame bundle such that

$$d\omega^i = \omega^i_j \wedge \omega^j$$

and $\omega^i_j = -\omega^j_i$.

Surfaces

Any surface in M can be represented by a local embedding $U : \mathbb{R}^2 \to \mathbb{R}^3$. We can lift any such embedding to an embedding

$$\phi : \mathbb{R}^2 \to FM.$$

Such a lift is the original embedding together with a choice of frame at each point of the surface. We can adapt such liftings to the geometry of the surface.
For any surface there is an embedding ϕ such that $\phi^*(\omega^3) = 0$. In more geometric terms: the tangent space to the surface is spanned at each point $U(x)$ by the first two basis vectors in the choice of framing. We call such lifts "first order adapted".

Integral surfaces

The idea is now to consider integral manifolds of a suitable exterior differential system. We want that $\omega^3 = 0$ on these surfaces. But then

$$0 = d\omega^3 = -\omega_3^1 \wedge \omega^1 - \omega_3^2 \wedge \omega^2.$$

It follows (Cartan's lemma) that

$$\omega_3^1 = a_1^1 \omega^1 + a_2^1 \omega_2$$
$$\omega_3^2 = a_2^1 \omega^1 + a_2^2 \omega_2$$

for certain functions a_j^i.

The exterior differential system

We add new coordinates a^i_j to the system and define the 1-forms

$$\begin{aligned}
\theta^0 &= \omega^3, \\
\theta^1 &= \omega^1_3 - a^1_1\omega^1 - a^1_2\omega^2, \\
\theta^2 &= \omega^2_3 - a^1_2\omega^1 - a^2_2\omega^2.
\end{aligned}$$

We will look for integral surfaces of the exterior differential system generated by θ^0, θ^1, θ^2 with independence condition $\omega^1 \wedge \omega^2 \neq 0$. These integral surfaces correspond to adapted embeddings of surfaces in $\mathbb{R}^3$.

Apply the Cartan-Kähler theorem

The structure equations are

$$d \begin{pmatrix} \theta^0 \\ \theta^1 \\ \theta^2 \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 \\ da_1^1 & da_2^1 \\ da_2^1 & da_2^2 \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix}.$$

- No torsion.
- Cartan characters: $s_1 = 2, s_2 = 1$.
- The general surface in $\mathbb{R}^3$ depends on one function of 2 variables.

Curvature conditions

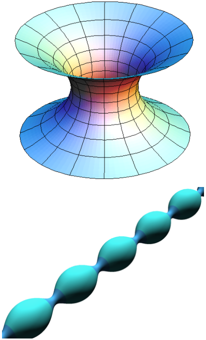
The coefficients $A = (a^i_j)$ represent the shape operator of the surface. In fact

$$\det A = a^1_1 a^2_2 - (a^1_2)^2 = \text{Gauss curvature}$$
$$\operatorname{tr} A = a^1_1 + a^2_2 = \text{mean curvature}$$

Let us construct surfaces with constant mean curvature (soap bubbles). We restrict our exterior differential system to the submanifold $\operatorname{tr} A = 0$, i.e. $a^2_2 = -a^1_1$.

The new structure equations are

$$d\begin{pmatrix}\theta^0\\ \theta^1\\ \theta^2\end{pmatrix} \equiv \begin{pmatrix}0 & 0\\ da_1^1 & da_2^1\\ da_2^1 & -da_1^1\end{pmatrix} \wedge \begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix}.$$



- No torsion.
- Cartan characters: $s_1 = 2, s_2 = 0$.
- The general surface in $\mathbb{R}^3$ with constant mean curvature depends on two functions of 1 variable.

Why (not) the Cartan-Kähler theorem?

- It is geometric, one does not need local coordinates. Consider for example the equation for constant mean curvature in local coordinates.

$$H = \frac{(1 + h_v^2 h_{uu} - 2h_u h_v h_{uv} + (1 + h_u^2)h_{vv}}{2(1 + h_u^2 + h_v^2)^{3/2}} = \text{constant}$$

- It is very powerfull: it can handle non-linear system of any order
- It is algorithmic: once your problem is formulated in differential forms one can start applying the methods
- It only works in the analytic setting
- The calculations can become very complicated
- The dimension count of the solutions might not give you insight into the problem. For example: we still have no idea how CMC surfaces look like.

Skip isometric embedding example

The isometric embedding problem

Let M be a manifold with a smooth metric $g = g_{ij} dx^i dx^j$. Can we isometrically embed this manifold into $\mathbb{R}^N$ for some N?
Let $x^1, \ldots, x^n$ be coordinates on M and let the embedding be given by $x \mapsto u(x) \in \mathbb{R}^N$. The condition that u is preserves the metric is

$$g_{ij} = \frac{\partial u}{\partial x^i} \cdot \frac{\partial u}{\partial x^j}.$$

Since g_{ij} is symmetric in i, j these are $n(n+1)/2$ equations for N unknown functions $u^1, \ldots, u^N$.
The general surface can not be embedded in R^N if $N < n(n+1)/2$.
For analytic metrics g the dimension $N = n(n+1)/2$ is precisely enough. We will prove this by using the Cartan-Kähler theorem.

Let (M, g) be a surface with analytic metric. Let FM be the orthogonal frame bundle of M, i.e. a point in FM is a pair (x, E) where $x \in M$ and E is an orthonormal frame in $T_x M$. We represent an orthonormal frame by an isomorphism $T_x M \to \mathbb{R}^2$. On FM we define the soldering form, or canonical form, by

$$\eta(X) = E \circ T\pi(X).$$

Note that η is a 1-form valued in $\mathbb{R}^2$. We choose a basis η^1, η^2.

Theorem

There are unique anti-symmetric 1-forms η^i_j on FM such that

$$d\eta^i = \eta^i_j \wedge \eta^j.$$

The structure equations for η^i_j are

$$d\eta^i_j = -\eta^i_k \wedge \eta^k_j + (1/2)R^i_{jkl}\eta^k \wedge \eta^l$$

where R is the Riemann curvature tensor of the metric g.

On the frame bundle of $\mathbb{R}^3$ we can introduce in the same way $\omega^1, \omega^2, \omega^3$ and 1-forms ω^i_j.

Embeddings

Every isometric embedding $M \to \mathbb{R}^3$ we can extend to an embedding $M \to FM \times F\mathbb{R}^3$. There are lots of different ways, but we will adapt the embedding to the *geometry* of the system. For every isometric embedding we can arrange that

- $\omega^3 = 0$
- $\eta^1 - \omega^1 = \eta^2 - \omega^2 = 0$

The converse is also true: integral manifolds in $FM \times F\mathbb{R}^3$ that satisfy the two conditions above together with the independence condition $\eta^1 \wedge \eta^2 = 0$ correspond to isometric embeddings.
So we have succeeded in translation our isometric embedding problem to a geometric problem in terms if 1-forms! Let's try to apply Cartan-Kähler!

Structure equations

The differential ideal is spanned by $\eta^1 - \omega^1$, $\eta^2 - \omega^2$, ω^3. The structure equations are

$$\begin{aligned} d\omega^3 &= -\omega^1_3 \wedge \eta^1 - \omega^2_3 \wedge \eta^2 \mod I \\ d(\eta^1 - \omega^1) &= -(\eta^1_2 - \omega^1_2) \wedge \eta^2 \mod I \\ d(\eta^2 - \omega^2) &= (\eta^1_2 - \omega^1_2) \wedge \eta^1 \mod I \end{aligned}$$

By Cartan's lemma it follows that we must have also $\eta^1_2 - \omega^1_2 = 0$. We add this condition to our ideal I and start again.

Structure equations again

The differential ideal is spanned by $\eta^1 - \omega^1$, $\eta^2 - \omega^2$, ω^3, $\eta^1_2 - \omega^1_2$. The structure equations are

$$d\omega^3 \equiv -\omega^1_3 \wedge \eta^1 - \omega^2_3 \wedge \eta^2 \mod \tilde{I}$$

$$d(\eta^1 - \omega^1) \equiv 0 \mod \tilde{I}$$

$$d(\eta^2 - \omega^2) \equiv 0 \mod \tilde{I}$$

$$d(\eta^1_2 - \omega^1_2) = \ldots \tilde{I}$$

The system is not in involution, so we prolong everything.

Prolongation

We add variables h^3_{ij} to the manifold. We add $\theta_i = \omega^3_i - h^3_{ij}\eta^j$ to $\tilde{I}$. The structure equations for $\eta^1_2 - \omega^1_2$ yield

$$d(\eta^1_2 - \omega^1_2) = (h^3_{11}h^3_{22} - h^3_{12}h^3_{21} + R^1_{212})\eta^1 \wedge \eta^2.$$

There is torsion so we must restrict to the codimension 1 submanifold defined by $h^3_{11}h^3_{22} - h^3_{12}h^3_{21} + R^1_{212} = 0$. Whenever $R \neq 0$ this is a smooth submanifold of $FM \times F\mathbb{R}^3 \times \mathbb{R}^3$.

Final structure equations

With suitable π_1, π_2 (see [Ivey and Landsberg(2003), p. 172–173])

$$
\begin{aligned}
d\omega^3 &\equiv 0 \mod \tilde{I} \\
d(\eta^1 - \omega^1) &\equiv 0 \mod \tilde{I} \\
d(\eta^2 - \omega^2) &\equiv 0 \mod \tilde{I} \\
d(\eta^1_2 - \omega^1_2) &\equiv 0 \mod \tilde{I} \\
d\begin{pmatrix}\theta_1 \\ \theta_2\end{pmatrix} &\equiv \begin{pmatrix}\pi_1 & \pi_2 \\ \pi_2 & (h_{22}/h_{11})\pi_1 - 2(h_{12}/h_{11})\pi_2\end{pmatrix} \wedge \begin{pmatrix}\eta_1 \\ \eta_2\end{pmatrix}
\end{aligned}
$$

The Cartan characters are $s_1 = 2, s_2 = 0$. The general isometric embeddings exists and depends on two functions of 1 variable.

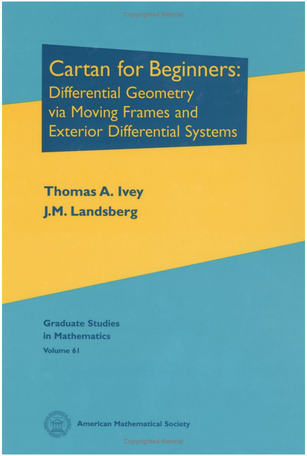
The C^∞ problem

The theorem is not true near points where the curvature is degenerate. However

Theorem (Gromov)

Every surface with C^∞ metric embeds isometrically in $\mathbb{R}^5$.

Want to know more?



or

BCG^3 book

- R. L. Bryant, S. S. Chern, R. B. Gardner, H. L. Goldschmidt, and P. A. Griffiths.
  Exterior differential systems, volume 18 of *Mathematical Sciences Research Institute Publications*.
  Springer-Verlag, New York, 1991.
  ISBN 0-387-97411-3.
- Thomas A. Ivey and J. M. Landsberg.
  Cartan for beginners: differential geometry via moving frames and exterior differential systems, volume 61 of *Graduate Studies in Mathematics*.
  American Mathematical Society, Providence, RI, 2003.
  ISBN 0-8218-3375-8.